The Mathematics Teacher

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Mathematics, missiles, and legislation[1]

STEWART SCOTT CAIRNS, *University of Illinois, Urbana, Illinois.*

While our schools need to fight anti-intellectualism, we still have faith that, "In matters of intellect as well, the products of a free educational system will prove superior in the long run to those of a system dedicated to the efficient production of servants of the state or of a particular political and economic order."

NOT LONG AGO, I encountered the term "fortunate deviate" applied to the superior student. A seven-foot basketball player may be a great athlete, but a student of high intellectual attainments is a deviate. I don't know who coined this unhappy phrase. It sounds like the brain child of a dabbler in educational statistics. If he believes the term devoid of unpleasant connotations, let him call an exceedingly beautiful woman a gorgeous deviate and learn his lesson the hard way.

At the other end of the scale the inferior student is conventionally referred to as "exceptional." When I was young, I would have assigned a favorable connotation to "exceptional student," meaning "exceptionally good," but this just proves that I'm now young like Jack Benny, for the term has long since been perverted in educational circles. Personally, however, I believe in speaking kindly of the slow learner. In that spirit, I'll even gulp down the distorted adjective "exceptional," but that is no reason for offending the gifted.

Our American culture is young in comparison with those of Europe and the Orient. We still cherish the qualities most precious to the pioneers who tamed the wilderness and to the revolutionaries who fought for their principles and our freedom until they won. We also admire, with occasional reservations, the pioneers of industrial expansion who amassed Gargantuan fortunes while developing the vast resources and the massive economy of our country. On the intellectual side, we admire the noble democratic philosophy formulated by our forefathers, especially because its conception and implementation required a rare mixture of brains and courage. And we highly esteem the famous Yankee ingenuity in its great range of manifestations, from the gadgets which give us the highest living standards of all time to the scientific and technological achievements which have won so exalted a position for American experimental research.

But the picture changes when we step into the world of purely intellectual achievements, devoid of all obvious or intended application to the development of material goods. This world embraces the realm of the humanities, of philosophy, and of pure mathematics. Not only are accomplishments in these fascinating and difficult areas held in low esteem, but their devotees are even viewed with a measure of suspicion. A hardened adult scholar, irrevocably committed to the intellectual life, can take this public attitude in his stride and can console himself

[1] An address delivered at the 1958 Annual Meeting of the National Council of Teachers of Mathematics in Cleveland, Ohio.

with the approbation of his like-minded colleagues, but an adolescent student, gifted or afflicted with intellectual tastes and talents, may well be deviated from his normal development by the attitude of his companions. Popular opinion to the contrary, a superior intellect is often housed in a superior body and is often associated with a well-developed gregarious personality sensitive to the social discrimination which the so-called "brain" is liable to encounter. He is therefore likely to turn his superior talents to those extracurricular activities which will win him a measure of glory and the cherished admiration of his associates.

Some Russian satellites have lately stimulated an incipient change in the American attitude toward academic subjects. Indeed, a professor of education recently remarked to me that he thought he saw signs of a wave of intellectualism hitting the schools. Knowing my predilections, the friendly fellow was trying to make me happy and, being good-natured myself, I accepted the remark in the intended spirit, even with some glee, for the very idea of intellectualism invading the schools is as delightful as the thought of a wave of religion sweeping through the churches. Any intellectual surge which does come will be a consequence of panic reaction to Sputnik, which drove the American public to shout for higher educational standards, especially in mathematics and science.

But "the tumult and the shouting dies." Our own satellites are wheeling about in space, and we may even beat the communists in the race to hit or circle the innocent moon. We can then slip back, part way at least, toward the happy-go-lucky schooling to which America is accustomed. The backsliding will be only partial, for those who have campaigned so long and patiently for higher scholastic standards have established a bridgehead. The public will not quickly forget the shock of realizing that a powerful adversary can match us satellite for satellite and bomb for bomb, can rival or outdo us in technical assistance to other countries and in the war of international propaganda. I fear, however, that this same public does not generally realize that the USSR started from almost nothing, as far as even literacy goes, somewhat over forty years ago, while we were well advanced toward our present status. If someone gave you a nine-mile head start in a race and overtook you one mile later, would you not expect him to shoot ahead?

The public is now convinced that an extremely and increasingly high level of scientific accomplishment is essential to our national security and possibly to our very survival. The public is also aware that educational reform is a necessary condition for the attainment of the vitally necessary level. So strong is this conviction that the ever-sensitive Washington politicians are leaping onto the bandwagon. When this happens we can indeed take heart, not because they will necessarily do something constructive, but because they are such a barometer of public opinion.

Despite the political straws in the wind, we should not naïvely conclude that Congress will (even if it could or should) legislate us into an educational millennium. There is, indeed, reason for skepticism over the prospects of the principal educational measures now on the floor of the Senate and the House. This is unfortunate, for these measures, though far from perfect, would do much more good than harm. It would be a shame to reject a great benefit just because one can imagine ways of making it even greater. If substantial educational legislation fails of adoption, it will not be primarily for so lofty a reason as a clash of opinions over the proper relative roles of federal and state governments in the field of education. A far more cogent cause will be that the mailbags of the legislators have been singularly lacking in supporting letters from those who should be most concerned. The mailbags are their barometers. As they are likely to conclude, this lack of letters may indeed signify a

lack of support for federal efforts to improve the schools. But a barometer can fool you. It may merely mean that those who favor educational bills are, by and large, not the kind to express their views in letters to legislators. In either case, the result is the same. Many a congressman will base his vote largely on his assessment of the opinions of his constituents.

It is true that the current legislation has been drawn up somewhat hastily, without a time-consuming, realistic analysis of needs. But it is desirable that some early action be taken, to be revised or supplemented in the light of a continuing analysis. The latter would involve an accumulation, organization, and interpretation of information (much of it already collected) as a guide to appropriate measures. It might clarify, though it could hardly settle, the thorny question of the proper role of the federal government in education.

It is no part of my present purpose to analyze the pending legislation, although I will comment on it in a general way from time to time. I first note the danger of reading into it, or into the accompanying publicity, an implication that the primary intention is to use scientific education for the development of more effective instruments of destruction. A cartoon several weeks ago showed someone telling a teacher to "teach them to get along, but teach them science so they can blow each other to bits if they don't."

Most of the recent public clamor for scientific training stems from motives as materialistic as you would find anywhere in the world. If you doubt this, reflect on a recent Gallup poll, which indicates that about 25 per cent of our high schools have already instituted post-Sputnik curricular changes and about another 25 per cent contemplate so doing. If it took Sputnik to stimulate these changes, one can hardly ascribe them to the motive of education for the enrichment of life and the ennobling of humanity. It is not a symptom of intellectualism to seek stronger schools for the sake of national defense or even for the peaceful purpose of still further advances in our scale of living.

By no means would I condemn or reject a valuable improvement merely because its motivation is less pure than one would expect from an ancient Greek philosopher. The ivory tower which I inhabit has a few doors and windows. I welcome these reforms and hope they are harbingers of more to come. But I stoutly maintain that the truest and broadest education, as contrasted with purposeful training, is the royal road to human welfare, and, on the way to that distant goal, it leads most directly to the protection and reinforcement of our democratic way of life. Experience through the ages amply reveals that pure philosophic speculation leads ultimately to practical political or social applications, and that science for its own sake provides a rich storehouse from which one can draw the knowledge essential to the taming of nature.

The man of action intervenes between the philosopher and the organization of the state. The practical scientist sees a way of bringing abstract theory back into the world of things and events. But the philosophy and the theory must be there, or progress will grind to a halt. If the Greeks had not, for purely cultural reasons, studied the geometric properties of the ellipse, how could Kepler have taken the giant step leading into our modern understanding of astronomy? And without the latter, who could dream of an artificial satellite? Behind our knowledge of fission and fusion, with all its military and peaceful potentialities, a long sequence of theoreticians stretch into the past. The most enlightened materialism is thus not materialism at all, but idealism, which has as by-products the greatest practical benefits for mankind. Like happiness, material progress comes less to those who make a specific goal of it, but is, rather, incidental to the pursuit of a higher purpose. This is admittedly a materialistic argument for idealism, but

maybe it is a way of easing people into it.

In the special case of mathematics, it is important to put across the fact that the applied mathematics of today was part of yesterday's pure mathematics and that part of today's pure mathematics (no one can say *what* part) is the applied mathematics of tomorrow.

In general, the political organizations of today grew from the philosophy and the political theorizing of our ancestors. For the sake of our descendants, we must cultivate and encourage speculative philosophy, literature, art, drama, scientific theorizing, and pure mathematics. From them will derive the political organizations, social customs, pleasures, and living standards of later generations. The nation which excels in them will chart the course of the future far more than the country with the greatest military might.

In this area, our democratic freedoms give us an advantage over the USSR. The latter, apparently realizing that some measure of liberty of thought is essential to scientific progress, has created an island of relative freedom for its scientists. With this priceless commodity added to material rewards, it is no wonder that there is a fierce, widespread desire in the Soviet Union to qualify for the scientific service of the state. But we, if we correct some of our major tactical blunders, should, with our great sea of freedom, forge far ahead.

The problem of a sufficient supply of highly qualified teachers is at the very heart of our current educational crisis, and our national anti-intellectualism underlies the teacher shortage. If the public set a high value on learning, the teaching profession would be so attractive that a much larger share of our best talent would flock into it. In working away at this deep, basic problem, we might commence with a reformation of those certification requirements which stress mere teacher training at the expense of genuine education. We might also correct the teachers' working conditions so that they will have time and energy for personal study. Let the bulk of their present noninstructional tasks be taken over by clerks and specially trained educational assistants.

The military motivation for educational reform reminds me that someone recently suggested a new TV program to be entitled "Have slide rule, will travel." Times have indeed changed since Wellington's dictum that the battle of Waterloo was won on the playing fields of Eton. The next war, if it comes and if anyone can be said to win it, will be won within the classrooms rather than outside on the playing fields.

Yet it would be tragically sad, and perhaps disastrous, if we deliberately directed our educational procedures toward military ends or even motivated them by their utility in a world at peace. The pleasure and the sportsmanship of the playing field could not, by any stretch of the imagination, be successfully developed as a conscious device for the production of good soldiers. Paradoxically, this motivation would defeat its own purpose. The regimented sports program of a dictatorship may produce athletes, but will not produce the spirit of loyalty and supreme endeavor which are by-products of the playing fields of a free nation. In matters of the intellect as well, the products of a free educational system will prove superior in the long run to those of a system dedicated to the efficient production of servants of the state or of a particular political and economic order.

I said "in the long run," and I remark that we may have to take drastic emergency action to ensure the very existence of a "long run" during which this superiority can be demonstrated. Expediency may drive us to temporary distortions of our curricula in the direction of science and technology, in order to preserve the possibility of a better-balanced program in some happier day. Thus, when a war comes, do the citizens of a free country willingly accept, as soldiers or civilians, a temporary curtailment of their precious

liberties. They do so knowing that victory will restore these liberties and that defeat may entail their complete sacrifice. We, as educators, should take deliberate account of the present threat to the democracy which makes a free educational system possible. While expediency may lead us into a temporary over-emphasis on some subjects at the expense of others, let us not lose sight of our educational ideals. There must be no permanent warping of our culture.

Meanwhile, for the sake of immediate as well as ultimate objectives, let us now bend our efforts to the creation of a climate in which intellectual achievement is honored, just as athletic prowess has long been recognized. Some starts have been made in this direction. In our own field of mathematics, I point to the statewide and national prize competitions sponsored by the Mathematical Association of America. Last Saturday, the local papers in my home town of Urbana, Illinois, carried, each of them on page 3, a good write-up of the third consecutive top-place victory of the University High School mathematics students in the Illinois contest. Of course, the big high school football game of the year, in its season, has the power to crowd even the juiciest murder out of the main headlines on page 1, but still this is progress, and it is the sort of thing I mean. Here were students who could read their names in the paper. They could feel a just pride in the competitive demonstration of their individual attainments and in the credit which they, as a sort of informal team, were bringing to their school. Theirs is like the satisfaction of athletes who have won a football championship and of the individual star who has made a 90-yard run for a touchdown. Remember that, in ancient times, poets as well as athletes and soldiers received crowns of laurel.

Another step in the direction of public recognition of academic achievement is in one of the two principal educational measures before Congress. The statement of purposes of the Hill-Elliott bill reads, in part: "to strengthen the national defense, advance the cause of peace, and assure the intellectual pre-eminence of the United States, especially in science and technology." This goal of "intellectual pre-eminence" is something new and refreshing on Capitol Hill. It is well to include it along with the more practical goals, even though urgency lends a predominating emphasis to the meeting of our critical national needs. Under this bill, scholarships are independent of financial status and are intended to recognize merit, in the hope of stimulating intellectual interest and achievement on a broad basis. Another related feature is an authorization for the Commissioner of Education to award to the top five per cent of high school graduates medals and scrolls bearing the inscription "Congressional citation for outstanding scholastic achievement." This looks like a good modern equivalent of a crown of laurels.

For reasons which I have never entirely fathomed, educators have long frowned on scholastic competition. They do so partly because of a fear that the less competent students will suffer frustration and other psychological ills if they discover that others learn more readily than they do. It is even considered undemocratic, by some queer interpretation of the concept, to give special attention to the gifted. They are expected to hold back to the average pace, with the result that both they and the country are partly cheated of the satisfaction and the benefits which are the proper fruits of their talents. The news account which I read of the Gallup poll on post-Sputnik changes in the high schools contained the following paragraph: "Other principals reported a change in the direction of grouping the more intelligent pupils together, but in most cases such a long-term policy shift has yet to 'get off the ground' and is still in the planning stage." Apparently this is a radical departure, although special attention to so-called "exceptional students" is a matter of general policy. The schools must

provide differentiation to permit the fullest development of the most talented. As a corollary, they must make appropriate provision for the average and those below average. It does no one very much good to send to engineering and liberal arts colleges large numbers of students who would find greater happiness and success in two-year technical institutes or junior colleges.

Let me finally turn to the question of who should make educational decisions, of what agencies should control the schools. Clearly, we are unwilling to let a child or his parents decide that he should not go to school. We occasionally read of parents being hailed into court for attempting to deprive their children of the legally required period of education. The newspapers in Cleveland two days ago carried accounts of such a case, involving religious considerations, where the sons of three Amish couples near Wooster, Ohio, had quit school after the eighth grade but before reaching Ohio's compulsory school-age of sixteen.

At the other extreme, I know of no one who would want to have the schools of the country controlled from Washington. Somewhere between a centralized national control and a completely individualized control we are, then, to find the happy medium. This is not primarily a matter of democratic principles, for a democratically-governed country can, and some of them do, place the schools under the supervision of a national minister of education. Without any real danger of federal control, it is possible at least to have a system of national scholarships, though the extent of our need for them is debatable. With such aid, there is some danger that students who are, in a sense, "sent" to school by the government will feel they have a claim for suitable employment afterwards. This attitude was not unknown among those who benefited by the GI bill. It would be even more likely among those sent for the special purpose of developing scientifically-trained manpower. There would indeed be some justification, since trained engineers and scientists must be appropriately used (by industry, government, or the educational system) to preserve their efficiency.

Another appropriate area of federal aid, to my mind, is in the matter of buildings and equipment, though I don't know where the lines should be drawn separating the local, state, and federal functions. In any case, this type of federal help is largely a pipe dream because of the segregation issue, the arguments over the danger of federal control, and the strong opposition in a number of states, especially those where more money would be likely to go out than to come in.

America is unique among comparable countries in the degree of decentralization of our educational controls. We have a strong tradition of local school boards, locally selected, making important decisions implemented through superintendents of schools. I should not like to lose this tradition with all its advantages, but I should like to see some modification. There now exist, as you know, varying degrees of state control of educational standards. I should like to see them all brought up to a certain level. I should like to see state laws requiring that every student have the opportunity, but of course not the obligation, of preparing for the best of our colleges. This would lend added significance to the requirement of universal school attendance.

To me it seems absurd to require every child to go to school, yet not to impose reasonable requirements on what the schools shall offer him.

Whatever be the cost, the nation cannot afford to permit local authorities to deny to students of their districts the privilege of studying those mathematical, scientific, and other academic subjects appropriate to the secondary schools. While we must do right by students not bound for college, we must take great care that no student with the ability and desire to profit by higher education is denied the opportunity to prepare adequately for it.

This includes, in mathematics, the opportunity to get ready in high school for a beginning college course combining analytic geometry with an introduction to the calculus.

I realize that there are mammoth problems involved in achieving this goal on a nationwide basis. Fortunately for me, I do not deem it the role of an after-dinner speaker to discuss such formidable issues in detail, or even to reel off the familiar roll call of major difficulties which confront us in the years ahead. Let us, instead, follow a fine banquet with a good night's sleep, and start nibbling away at our problems tomorrow.

What's new?

BOOKS

Secondary

Basic Geometry, George David Birkhoff, Ralph Beatley. New York: Chelsea Publishing Company (1958 reprint of 2d edition, 1941). Cloth, 294 pp., $3.95.

Elements of Plane Trigonometry, Henry Sharp, Jr. Englewood Cliffs: Prentice-Hall, Inc., 1958. Cloth, ix+274 pp., $4.95.

Plane Trigonometry (2d edition), John J. Corliss, Winifred V. Berglund. Boston: Houghton Mifflin Company, 1958. Cloth, xii+397 pp. $4.00.

College

A Modern Approach to Intermediate Algebra, Henry A. Patin. New York: G. P. Putnam's Sons, 1958. Cloth, viii+224 pp., $3.75.

An Introduction to Combinatorial Analysis, John Riordan. New York: John Wiley and Sons, Inc., 1958. Cloth, x+244 pp., $8.50.

An Introduction to the Theory of Functions of a Complex Variable (Dover republication), P. Rienes. New York: Dover Publications, Inc., 1957. Paper, x+552 pp., $2.75.

A Treatise on the Analytic Geometry of Three Dimensions, George Salmon. New York: Chelsea Publishing Company (1958 reprint of 5th edition, 1955). Cloth, xxiv+470 pp., $4.95.

Calculus, Walter Leighton. Boston: Allyn and Bacon, Inc., 1958. Cloth, x+373 pp., $6.95.

College Mathematics, Kaj L. Nielsen. New York: Barnes and Noble, Inc., 1958. Paper, xviii+302 pp., $1.95.

College Plane Geometry, Edwin M. Hemmerling. New York: John Wiley and Sons, Inc., 1958. Cloth, ix+310 pp., $4.95.

Differential Equations (Dover republication), Forest Ray Moulton. New York: Dover Publications, Inc., 1958. Paper, xv+395 pp., $2.00.

Elementary Coordinate Geometry (2nd ed.), E. A Maxwell. New York: Oxford University Press, 1958. Cloth, 336 pp., $4.00.

Elementary Differential Equations (2nd ed.), Earl D. Rainville. New York: The Macmillan Company, 1958. Cloth, xii+449 pp., $5.50.

Essential Business Mathematics (with teacher's manual and key to text) (3rd ed.), Llewellyn R. Snyder. New York: McGraw-Hill Book Company, 1958. Cloth, x+470 pp., $5.50.

Essential Mathematics for College Students, Francis J. Mueller. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958. Paper, xiii+238 pp., $3.95.

Functions of Complex Variables, Philip Franklin. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958. Cloth, ix+246 pp., $6.95.

Geometrical Drawing for Students, Richard Marriott. London: Methuen and Company, Ltd., 1958. Paper, 112 pp., 7*s.* 6*d.*

Handbook of Calculus, Difference and Differential Equations, Edward J. Cogan and Robert Z. Norman. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958. Cloth, xii+263 pp., $4.50.

Homology Theory on Algebraic Variables, Andrew H. Wallace. New York: Pergamon Press, 1958. Cloth, viii+115 pp., $5.50.

Intermediate Algebra for Colleges, Gordon Fuller. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1958. Cloth, vii+258 pp., $3.90.

Introduction to Difference Equations, Samuel Goldberg. New York: John Wiley and Sons, Inc., 1958. Cloth, xii+260 pp., $6.75.

Introduction to the Theory of Sets, Joseph Breuer (translated by Howard F. Fehr). Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958. Cloth, viii+108 pp., $4.25.

Mathematical Foundations of Information Theory, A. I. Khinchin (translated by R. A. Silverman and M. D. Friedman). New York: Dover Publications, Inc., 1957. Paper, 120 pp., $1.35.

Computations with approximate numbers[1]

D. B. DELURY, *Ontario Research Foundation, Ontario, Canada.*

Contrary to present practices in the schools, there are no simple answers and no general rules for computing with approximate numbers.

THERE IS ROOM, I think, for the view that it is improper to speak at all of "approximate numbers." Admittedly this is jargon, an abbreviated phrase that we use to spare us the effort of saying over and over exactly what we mean, namely, "a number whose value approximates the value of this or that magnitude." There is no denying the usefulness of jargon. As long as it is used within the circle of experts or specialists for whom it is intended, there is probably little danger of confusion. On the other hand, there is no doubt that it can cause serious misunderstandings when it is used outside these circles. We have all seen the weird interpretations put on such words as infinity, curved space, the fourth dimension. It seems to me that "approximate number" is especially exposed to misinterpretation, because we have taken the word "approximate" from the place where it belongs, as a description of the *process* that produced the number, and attached it to the number itself.

I incline to the view that those of us who are concerned with giving accurate instruction should avoid this kind of jargon or, at least, delay its introduction until the need for it and the precise meaning to be attached to it have become amply clear.

When we come to speak about computations with such numbers and more particularly about the proprieties concerning the presentation of the results of these computations, it is, I believe, important to distinguish sharply between the two ways in which questions of approximation can arise:

1. through arithmetical operations only;
2. through measurement.

For the present, I shall speak of 1.

Since calculations for the most part are conducted with digital numbers, we are compelled to use approximations to irrational numbers and usually we find it convenient to use decimal approximations to rational numbers also. Here there ought to be general agreement on the procedures to be followed in truncating numbers, not because there is only one way or because there is necessarily one best way, but simply because, if only one rule is used, it need not be stated over and over.

If, for example, I wish to use a modest approximation to e, I look up its value in a book and find $e = 2.718281 \cdots$. Now I could truncate it simply by chopping it off at the desired place, e.g., $e = 2.71$. Then, I am required to take the view that e lies somewhere in the range 2.71–2.72. Let us make no mistake here; these numbers are exact and should be so treated. They simply mark the boundaries of a range within which *e certainly lies*. If I want to make calculations with the value of e so specified and to know within what range

[1] An address delivered at the 1958 Annual Easter Meeting of the Mathematics and Physics Section of the Ontario Education Association. It is being reprinted with the permission of the *Canadian School*.

the true answer certainly lies, I must make the calculation with each end of the range, thus obtaining the range within which the true answer must certainly lie. These calculations should be exact, to the extent that they can be, unless we wish to add arithmetical mistakes to the uncertainty caused by the truncation. Thus e^2 lies in the range $(2.71)^2$–$(2.72)^2 = 7.3441$–7.3984.

The notion of a range within which a number must certainly lie is the foundation on which every accurate procedure for computation must be based. Indeed, some writers use the term "range number" to give the idea a name. Range numbers need not arise only through the kind of truncation I used in the little illustration. I suppose there are all sorts of ways in which we could come by the information that "this magnitude certainly lies within the range so and so." If we were prepared to use range numbers always and perform the dual calculations they require, there would be no need here to go any further, beyond developing some simple rules to assist in performing quickly the fundamental arithmetical operations with range numbers. The fact is, however, that rarely are we willing to do the work that is required to make a calculation with range numbers. We prefer to sacrifice both accuracy and clarity in order to be able to calculate with single numbers. In the example we have been using, we could accomplish this by replacing the range number by the value at the middle of the range, 2.715, thereby rendering the maximum inaccuracy as small as possible. Then $e^2 = (2.715)^2 = 7.371225$. One might complain that this answer is wrong. I would prefer to say that it is not complete. To make it complete, we would have to write

$$e = 2.715 \pm .005,$$

whence

$$\begin{aligned} e^2 &= (2.715)^2 \pm 2(.005)(2.715) + (.005)^2 \\ &= 7.371225 \pm .02715 + .000025. \end{aligned}$$

Hence e^2 lies in the range

$$7.344100\text{–}7.398400.$$

This is, of course, just another way of using the notion of range. The result is, and must be, the same whichever way we choose to write our numbers. In particular, the zeros that came out in this calculation may be retained or dropped, at our pleasure. We may remark, too, that even though one of these calculations uses more digits than the other, their accuracies are the same. Numbers written in the form —$\pm$— are sometimes called approximation-error numbers.

This second calculation is somewhat more tedious than the first, because the numbers we had to use require more nonzero digits. This stems from the way in which we performed the truncation. It gave us nice end points, 2.71 and 2.72, but a nasty mid-point, 2.715. If we had intended to use the second form of computation, we would have done better to use a form of truncation which requires as few digits as possible for the chosen accuracy, for the mid-point. We would, presumably, have said "e lies in the range 2.715–2.725," which has, it is true, nasty end-points but a nice mid-point, 2.72. We would then write e as the approximation-error number $2.72 \pm .005$. The largest possible inaccuracy is the same as in the first truncation.

It appears then, that if we propose to do our arithmetic with range numbers, we would use the first way of performing the truncation, and if we want to use the approximation-error form we would use the second.

Now in fact, we do not, as a rule, want to do our arithmetic in either of these ways. We want to calculate with the midpoints and forget about the inaccuracy. This point of view has dictated the universal practice of truncating in the second way, rather than the first. This process we call "rounding-off" and the resulting number we call a "significant number." A significant number, then, is one whose maxi-

mum inaccuracy is $\pm\frac{1}{2}$ in the last recorded digit. Significant numbers have the property that, for a given range of inaccuracy, they require fewer digits to specify exactly the mid-point than one would get by any other way of performing the truncation.

Since the convention about rounding off is, for all practical purposes, universal, it is generally understood that a truncated number is a significant number, and hence it is not necessary to specify the range of inaccuracy, since it is known to be $\pm\frac{1}{2}$ in the last recorded digit. This has some advantages. We know, for example, that the numbers listed in our tabulations of values of functions are significant numbers. On the other hand, the notion of significant numbers probably played no part in the computations that produced them. Significant numbers are not good numbers to calculate with, because the result of any computation with a significant number is not a significant number. For example, $e = 2.72$ is a significant number, i.e., e certainly lies in the range $2.72 \pm .005$. Hence, we can calculate that e^2 certainly lies within the range $(2.715)^2$–$(2.725)^2 = 7.371225$–7.425625. There is no significant number equivalent to this range number. The best we can do is 7.4, which yields a range 7.35–7.45. Now it is true that this range certainly contains e^2, but it is far wider than it has to be in order to have this property.

This kind of mistake snowballs rapidly in extended calculations. For this reason, people who take their computations seriously do not use significant numbers, nor do they necessarily state the results as significant numbers. Hence, today we never know whether the numbers we see are significant numbers or not, except in books of tables. There is no way of telling, from the look of a number, whether it is a significant number or not. Things are, to put it mildly, somewhat confused. There is really no need for this situation, either. Largely it springs from the attempt to use significant numbers in dealing with measurements about which I shall speak later. If we stay for the moment within the field of arithmetic, which is the only place where the notion of significant number has any meaning, there is no need whatever for any confusion.

Let us agree that any number that has been reached solely through the operations of arithmetic, if it cannot conveniently be stated exactly, *ought* to be written as a significant number if possible; if not, there should be an explicit statement of its accuracy.

Presumably, before we embark on such a calculation, we know how many digits we wish to have in the result. The only question, then, that needs answering is "How shall we carry out the calculation in order to get the result we want?" It appears that we have been asking our question the wrong way round. We have asked "How should we calculate with approximate numbers?" when it would have been better to ask "How should we proceed to obtain an approximate number with the desired degree of accuracy?"

Having asked our question thus, it must be admitted that there is no simple answer and there are no general rules. We can, however, easily trace through the effects of arithmetical inaccuracy, caused by truncation of numbers, in a single application of each of the fundamental operations of arithmetic, and these effects can, and have been, formulated as rules.

Perhaps I should stop here for a moment to gather up the substance of what I have been saying.

1. I have been talking only about arithmetic.
2. Exact computations with approximate numbers are quite feasible, if somewhat distasteful, using range numbers. Obviously such calculations can lead to numbers with many digits. There is nothing improper in this. Indeed, the number of digits has nothing to do with the question of accuracy. However, these numerous digits can be a nuisance and we might well adopt the position

that, in view of the range, perhaps large, within which the answer lies, it is not important that we know exactly the boundaries of this range. We would, therefore, round off our numbers at some chosen number of *decimal places*. The choice would be determined by the use to which our answer is to be put, not by any considerations of inherent accuracy.

3. A significant number is, by definition, one that cannot differ from the one to which it approximates, by more than $\frac{1}{2}$ in the last recorded digit. Significant numbers are formed by the process of truncation called rounding-off. Significant numbers are, or if you prefer, easily lead to, range numbers, but range numbers can rarely be written as significant numbers. The convention that numbers which result from *arithmetical operations only* be written as significant numbers is useful and should be maintained quite generally. It may be remarked, though, that these circumstances do not arise as often as might be supposed. Most of our calculations are made with numbers that we get from measurements and to these the notion of significant number does not apply.
4. Significant numbers are not well suited to numerical calculations and are not so used by people who take their arithmetic seriously.

In connection with this last item, the following quotations are pertinent:

> The loss in the number of significant figures in products and quotients, for example, is due not so much to accumulation of errors as to the simplicity that has been gained at the expense of precision.

By *simplicity*, here, must be meant *ease of computation*. Actually, the use of significant numbers in computation makes for considerable complexity. Here, for example, is the theorem for products and quotients of significant numbers.

> The product or quotient of two numbers, each containing n significant figures (at least two of which are not zero) is a significant number of at least $(n-2)$ figures. If the leading digits of these numbers are both equal to or greater than 2, then the product or quotient has at least $n-1$ significant figures.[2]

This, I suggest, is not a step in the direction of simplicity.

> The selection of a suitable type of approximate number depends on the purpose of the computation. Operations with significant numbers are easier and simpler than the corresponding operations with range or approximation-error numbers. They are quite satisfactory when additions, subtractions or a single multiplication or division are involved. They are also satisfactory when we are not concerned with the loss of significant figures in each operation. In most computational work we cannot afford this luxury.[3]

We can take it, then, that whenever we can avoid the use of range or approximation-error numbers, we will do so, but we prefer not to use significant numbers. What, then, do we do? One more quotation from the same book:

> For the basic linear problems the method of the next paragraph is to be preferred.
>
> An alternative method is the use of *incomplete* numbers. An incomplete number is an approximation-error number in which the error term is omitted. These numbers look very much like significant numbers, but, unlike significant numbers, the results may be recorded to any desired number of places. This method makes for ease with a machine, since all numbers to be placed on the machine may be rounded off to the same number of places. It must be remembered that any recorded number is not necessarily a significant number in the technical sense, that is, we do not know what the bound for the error may be.[4]

This is surely a curious statement, with its invention of the term *incomplete* number. It means simply this, that every number that enters into the calculation is treated as if it were exact, to a chosen number of decimal places. The number of decimal places is chosen to be adequate, or usually much more than adequate, to yield results of the required accuracy. This is the way computers usually do their arithmetic.

[2] Dwyer, *Linear Computations* (1951), p. 15.
[3] *Ibid.*, p. 33.
[4] *Ibid.*, p. 34.

Let us pass on now to the second way in which we come to deal with numbers that are approximations to others—those which arise from measurement. Let us restrict ourselves to physical measurements—the so-called measurements of psychology and such fields pose somewhat special problems.

To start the discussion on this topic, I offer you a quotation from N. R. Campbell: "Probably more nonsense is talked about measurement than about any other part of physics."

I acquired this quotation second-hand and I cannot say that he had in mind the kind of thing I must talk about today. If he didn't, I shall make the same statement in this context.

In the first place, measurement does not produce numbers. The result of a measurement should properly be stated in the form of a range, within which some point, line, or whatever is observed to lie. Presumably this process could be carried out in such a way that this range could be expressed as a number, plus or minus $\frac{1}{2}$ in the last recorded digit. This is not always done, by any means, but let us say that it is. Then, it seems to be an easy step to say that these recorded numbers, obtained from measurement, are *significant* numbers, that is, the numerical value of the magnitude, which our process of measurement is supposed to estimate, *certainly lies* within the range defined by $\pm\frac{1}{2}$ in the last recorded digit.

It is a pity that this is not true. If it were, the world of science would be much simpler than it is. The fact is, of course, as we all know, measurements don't behave this way at all. We all know, for example, that the real error of any measurement is not composed wholly of the error committed in making a final scale reading. Not uncommonly, two attempts at measuring the same thing produce two ranges which do not overlap at all and we cannot, surely, be certain that the true value lies in both of them.

This implies that the real error is likely to be far greater than that implied by the coarseness of the scale with which the final reading is made. Often it is so great that this final contribution to the error may be ignored. On the other hand, repeated determinations, made with even a coarse scale, can lead to an estimate which is far more precise than the coarseness of the scale would lead one to expect. What I am saying, then, is simply this: no number, significant, range, or any other, can, by itself, say anything about the precision or accuracy of the measuring process used to obtain it.

Now, this has been known for a long time. Over 100 years ago, Gauss and Laplace worked over this ground and produced the theory of errors, based on the notion of the frequency distribution, which is simply an idealization of the experience of people who make measurements on how errors of measurement behave. They did the job pretty well, too, and this theory has come down to us virtually unchanged, except that a comparatively new discipline, the design of experiments, has given it considerably greater depth and scope.

In any event, the theory of errors provides us with the only usable tool we have for dealing with errors of measurement. This tells us, among other things, that the treatment of errors cannot be undertaken as an exercise in arithmetic. The basic requirement is that the program of measurement be so arranged as to permit the estimation of a *standard deviation*, in terms of which the precision of the measuring process can be stated. The statement that the length of something or other is 11.3 inches, to the nearest tenth-inch, is either trivial or not true. The statement that the length of something or other is 11.3 inches, with a standard deviation of .2 inch, *is* a meaningful statement. It implies, among other things, that the "true" length is almost certainly somewhere between 10.7 and 11.9 inches, leaving aside the possibility of bias or systematic error, which is irrelevant to our

topic. Note, however, that this is *not* a range number, because we cannot know that the true value is *certainly* in this range. To say the same things in another way, there is no place, in connection with errors of measurement, for any talk about *possible* errors, meaning the maximum error that could possibly be encountered. The notion of possible error, or, I would prefer to say, possible *inaccuracy*, has meaning only in arithmetic.

I do not know how the notion has crept in that the theory of errors can be replaced by an exercise in arithmetic, coupled to a convention about the form in which the answer should be written, but I favor the view that the physicist is the culprit, not because I know of anything in the literature that points a finger at him, but rather because physicists have generally been loath to carry out their programs of observations in such a way that their real errors can be estimated and the theory of errors brought properly into play. In these circumstances, then, they have tried to assign what they call a "limit of error," that is to say, a maximum possible error. Let me quote from a book recently published, written by a physicist, on the Theory of Error:

> Many observers estimate the limit of error, the maximum amount by which the quantity may be supposed to be in error. Other observers believe such a procedure too conservative, since large errors are relatively improbable compared with small ones. Therefore, instead of using the full estimated value of the limit of error, these observers reduce it, perhaps by one-third. Since these are matters of opinion, no firm rules can be given and each experimenter must use his own judgment.[5]

This says to us, it seems, "Pick the error you like best."

Now, the theory of errors is not a matter of opinion and there is no doubt at all about how an experimenter should carry out his work in order to make proper use of it. This kind of humbug is an attempt to gain an air of respectability by sneaking under the mantle of the theory of errors without doing the work that it demands. With respect to practices of the kind revealed in this quotation, a remark made by Bertrand Russell in his *Introduction to Mathematical Philosophy* seems to me to be wholly pertinent.

> The method of postulating what we want has many advantages; they are the same as the advantages of theft over honest toil.[6]

Presumably this is not the place for an exhortation about the importance of the theory of errors. My whole concern here is to bring out one fact, that errors of measurement have nothing whatever in common with so-called approximate numbers. They are conceptually wholly different. One is concerned entirely with a question in arithmetic, specifically, with the magnitude of the mistakes (rather than errors) that can enter into an arithmetical calculation through truncation of numbers. The other has to do with the physical processes of making measurements and is based on empirical evidence about the way measurements behave. In the one, it is wholly proper to speak of the largest possible *mistake* introduced into an arithmetical calculation by truncation of numbers; in the other, it is quite improper to speak of a maximum possible *error* in a measurement. It is important, if we want to think accurately about these matters, that we keep these two things sharply separated.

Where, then, does all this leave us? Specifically, what does this mean for us, who are concerned with providing instruction that is accurate and comprehensible at the level at which we must give it?

I would not undertake to give any comprehensive answer to this question. Indeed, I believe that for me or anyone else to do so, without having tried and experimented a bit to build up some experience on what can be done successfully and what can't, would be sheer folly. On the

[5] Yardley Beers, *Introduction to the Theory of Error* (Cambridge, Mass.: Addison-Wesley Publishing Company, Inc., 1953).

[6] Bertrand Russell, *Introduction to Mathematical Philosophy* (London: George Allen and Unwin, Ltd., 1919).

other hand, there are certain principles which must be rigorously observed in *any* attempts in this direction. These have, for most part, already emerged, but I shall list them again and offer some opinions on the implications of these principles.

1. Errors of measurement should be sharply distinguished from mistakes in arithmetic of the sort that lead us to speak of significant numbers. I believe that this separation should be so stoutly maintained that no discussion of measurement would be permitted in mathematical subjects. In these, and especially in trigonometry, we have a good opportunity to see the way in which arithmetical calculations are properly carried out, with due attention paid to orderly procedures, mistakes introduced by truncation, and checks against blunders. After all, there is only one reason why anybody performs an arithmetical calculation. It is to get the right answer. As far as the question of approximate numbers is concerned, there is room here, I should think, for studying how truncation mistakes are propagated through simple calculations. I would make no room at all for horrible examples in which numbers with grossly different accuracies have to be combined. If the arithmetic is handled competently, this should not be allowed to happen. The emphasis should be on how one should carry out his calculation to arrive at a result of the required accuracy. It is, in my opinion, quite proper to put a problem in the following form: Two sides of a triangle are 9 metres and 12.3 metres long. The angle between them is 27°39′. Calculate approximately the area of the triangle and give a range within which the area of the triangle must lie.

The numbers given in this question are, by implication, exact.

It is true that there are many answers to this question, all of them correct. This is as it should be. If we want to single out a particular one of these, we can require the answer to so many significant figures or, what comes to the same thing, we can ask for the area, in square metres, correct to so many places of decimals.

One might object that the question so stated is artificial. If, by this, is meant that if one had to measure these quantities, he could not ask such a question, I agree. On the other hand, it is a real *mathematical* question and I remind you that I am talking about a course in mathematics. Furthermore, the question is no less artificial, from the point of view of measurement, if it is asserted that the sides are measured to the nearest centimetre and the angle to the nearest minute, or if the same notion is conveyed by adopting the convention of significant numbers and writing 9.0 instead of 9, and so on. It is, I repeat, no less remote from reality and, in addition, it carries the implication that we can cope with errors of measurement in this manner, which is monstrous. If we allow ourselves to go this far, we might as well go all the way and write our lengths as 9.0 and 12, because now they have the *same number* of *significant figures*. At this point, we are as far from the realities of measurement as we can get.

If we want to ask such a question in the only proper way it can be asked, when the dimensions of the triangle are measured, it would have to read somewhat as follows: The sides of a triangle are 9 and 12.3 metres, with a standard error of .12 metres; the included angle is 27°39′, with a standard error of 5′. Calculate the area of the triangle and its standard error.

This is no mean problem!

2. The basic notion in any discussion of mistakes in arithmetic, caused by truncation, is the range number. It is so simple a concept that any child can see all that is involved and it seems to me the natural place to begin. The significant number then emerges as a special kind of range number and its merits and weaknesses are immediately obvious. The dogma of the significant number has been with us for a long time and I fear it will plague us for some time to come, so we can hardly avoid discussing it in our teaching. However, let

us teach it as it is, not with a halo of false implications surrounding it, and we should make it clear that we do not encounter genuine significant numbers very often.

All in all, it seems to me that what needs doing and a general outline of how to do it are fairly clear, as far as purely arithmetical questions are concerned. It may be granted that the going gets rough when we come to elaborate calculations, such as the solution of large systems of equations, but such questions should not arise at an elementary level.

3. Things are less clear when we turn to errors of measurement, but it seems natural to me to suppose that the place to talk about measurements is where measurements are made, at least where they are made seriously, to learn something about a physical or chemical system. I do not regard as acceptable here the sort of exercise that has gained favor in some places, for example, sending a number of pupils with yardsticks to measure the length of a room, taking their measurements, which may read 24′, 23′9″, 23′7$\frac{1}{4}$″, and so on, then using these numbers to show how to reduce them all to the level of the worst, average them, and come out with an estimate of the length of the room. I can see nothing but harm in such exercises. Not only do they invoke the confusion of approximate numbers with errors of measurement, but they cast a false light on the process of measurement itself. In all honesty, anyone who treats numbers so obtained as anything but garbage makes a mockery of the whole idea of measurement. The making of measurements is too serious a business to be treated so casually. It is, among other things, a complex physiological and psychological process and is without meaning until stability and control have been demonstrated.

Really, I see nothing to be gained by talking about errors of measurement, except to people who make measurements, that is, in courses in physics, chemistry, and perhaps biology. Maybe laboratory work can be planned to give some indication of the way measurements behave. Measurements must be repeated and, indeed, if one is concerned to know the whole of his error, whole experiments must be repeated.

I do not suggest that it is desirable or feasible to introduce any discussion of the theory of errors at the high school level. It is a topic that apparently demands considerable maturity. Even in universities, no serious attempts are made to provide the rudiments of the theory of errors to all the people who need them. However, it should be quite feasible to discuss the notion of bias (the systematic error of the physicist) and that of true errors, which tend to compensate. In connection with these, it is vitally important to give a careful and thoughtful discussion of the notion of average, what it can accomplish and what it can't, when one has the right to use an average and when he hasn't. I believe that, today, the average is among the most overworked and most abused of all our concepts.

4. Above all, we must scotch the notion that the precision of an average or of a single measurement can be judged from the way in which it is written. Apparently the opinion is current that some impropriety attaches to running an average out to "more places than are warranted," indeed, that there is some suggestion of deception in that more precision is claimed than can be justified. Now, we may grant that running averages of measurements out to many places of decimals is silly, but the real impropriety lies not in this, but in failing to provide a standard error to go with this average.

Perhaps an example will gather up some of the notions I have been putting forward. The example comes unchanged from the A.S.T.M. *Manual on Presentation of Data.* (Incidentally, in this manual, which is wholly concerned with the presentation and interpretation of measurements, there is no mention of significant numbers or significant digits.)

The ten numbers presented in the table are the measured breaking-strengths, in pounds, of 10 samples of copper wire, taken to estimate the mean breaking-strength of wire from one production lot.

Specimens	Breaking-Strength $= X$	X^2
1	578	334,084
2	572	327,184
3	570	
4	568	
5	572	
6	570	
7	570	
8	572	
9	596	
10	584	341,056
	$5,752 = \Sigma X$	$3,309,232 = \Sigma X^2$

Average $= \Sigma X/10 = 575.2 = \bar{\bar{X}}$
$\Sigma X^2/10 = 330,923.2$
$(\bar{X})^2 = 330,855.04$
Subtract 68.16
Extract square root 8.26 = standard deviation.

Remarks

1. The numbers X may represent readings rounded off to the nearest unit, i.e., $\pm\frac{1}{2}$ in the last digit, but on the other hand they may not and without knowing the details of the measuring process it would be improper to assume that they do. (The fact that they all are even makes one wonder!) In any event, this is irrelevant. We will do the same things with these numbers, no matter how the readings were made.

2. The object in making these measurements is to estimate the mean breaking-strength of this lot of wire. The average of the observations is calculated as an estimator of this number. It is here calculated to one place more than those given in the data. This is in accord with A.S.T.M. recommended practice. There is positively no implication that the true mean lies between 575.15 and 575.25. A range within which it is *likely* to lie is calculable from the standard deviation.

3. In the calculation of the standard deviation (and indeed in the calculation of the mean also) the observed numbers are treated as exact. To do anything else would bring us out with no meaningful numbers whatever.

4. The standard deviation is given to two places more than the data. This also is recommended A.S.T.M. practice.

Let us look at these A.S.T.M. recommendations as they apply to this particular example. A small statistical calculation indicates that there is near-certainty that the interval 575.2 ± 8.94 straddles the true mean. In view of the width of this interval, it is likely that these decimals serve no useful purpose and 575 ± 9 would meet all the needs we have. Thus it appears that the A.S.T.M. rule has given us more places than we have any use for. This has happened because most of the error has come from sources other than the final scale reading. On the other hand, if most of our error *had* been so caused, as it might be if we were measuring the value of some physical constant, these decimals might be well worth having. We see, then, I think, the meaning of these rules. They are simple conventions which have been adopted to keep people from doing outrageously silly things. They have not been derived from any fundamental considerations. Furthermore, there could be circumstances, I think, in which I would choose not to follow them.

These considerations have a bearing on the advice we should give to students who make measurements in the laboratory and then make calculations with them. I assume that the laboratory work will not be carried out in such a way that the theory of errors can be applied and that the theory of errors will not have been taught anyway. In these circumstances, the error in the measurements and in the final calculated estimates cannot be known. Two questions require answers. How shall they carry out their arithmetic and how shall they present the results of their calculations?

The answers are easily given. In their arithmetic, all numbers are to be treated as exact, with the proviso that if the num-

ber of decimal places becomes unduly large, some of them may be eliminated by rounding off in the course of the calculation. The final answer should be rounded off to a reasonable number of decimals.

I am sure we would, all of us, like to have something more definite than this, but the fact is that there are no grounds for definiteness. And, after all, what difference does it make if one person runs his calculation out to, say, two more decimal places than another? As long as they have not stopped too soon, the basis for a preference between them is largely aesthetic. Of course, before we can adopt this indulgent view, we must get rid of the notion, to which we never had any right, that we are dealing with significant numbers. It is this notion that has led to the view that too many decimals in the answer imply deception. As far as deception is concerned, the shoe is entirely on the other foot. The contention that the result of a calculation, with numbers obtained from measurement, is to be interpreted as a significant number is practically certain to deceive. Let us look back at the example and, for purposes of illustration, treat the numbers as significant numbers. Then we certainly have the right to express the average of them as a significant number, to the same number of digits, i.e., 575. Then, since 575 is a significant number, the "true" value certainly lies in the range 574.5–575.5. This statement is simply not true.

To sum up, then, let us keep the significant number where it belongs, as a convenient convention for writing answers in pure arithmetic. It has no other use.

Wherever they send us to school

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Let's look at the new mathematics and science teachers

RAY C. MAUL, *Research Division, National Education Association.*

It is important that we keep an eye on trends in the supply of mathematics and science teachers. Teachers should encourage able students to prepare for the profession.

OCTOBER 4, 1957, is a date long to be remembered by every person engaged in operating the American educational system. That day marks a turning point in the attitude of the great body of citizens toward their schools—elementary, secondary, and higher. And in the foreground, literally in the spotlight, stand the mathematics and the science teachers.

The launching of an earth satellite was not needed to alert teachers and other educational workers to the rapidly changing demands of our society. New problems created by the upsurge of need for educated manpower in our fast-growing technology had already been recognized and vigorously attacked by the schools at all levels. The post-1950 concept of national defense, with its manpower implications, was seen to create a new demand for the services of persons who would ordinarily enter or continue in the teaching profession. And as will be shown later in this article, the expanded effort to identify and to stimulate the youth with talents for scientific studies can be traced back to the opening of the current decade.

But the spectacular action of the Russians last October 4 brought about far-reaching changes in the attitude of the general public toward the schools. Educational experts popped up everywhere. Persons who had achieved success in such widely different fields as building a submarine and writing a best-selling novel came forth with glib answers to complex problems. Some suggestions merit thoughtful consideration, but blanket condemnation by those who have freely mixed fact with fancy has only made more difficult the job of the thousands of dedicated teachers. Caricaturing the American high school as a carnival ranks high among the best ways to lead the public away from an understanding of its stake in education.

After some months of emotional turmoil the results of sober thinking are beginning to emerge. Before us lies a growing opportunity to interpret the successes, shortcomings, and unmet needs of the educational program at every level and in every locality. Before us also is the opportunity to co-ordinate more closely the efforts of the elementary with the secondary schools, and on through the institutions of higher education.

Paradoxically, the elementary schools have attained their greatest achievements during the past ten years, while the high schools have suffered their keenest competition for competent teachers, particularly in mathematics and science. The typical elementary school teacher now has one full year of college education beyond that of her counterpart in 1948. Also, the program of preparing elementary school teachers has steadily expanded its

emphasis on "general education" in the basic fields of knowledge. High school teachers will observe steady improvement in the ability of entering students to cope with the requirements of mathematics and science.

But what about the high school teaching staffs? How have they fared during the past decade in the search for adequate numbers of competent teachers? The supply-demand studies of the NEA Research Division reveal many interesting facts, even though some of them are not encouraging.

Factors in teacher supply

The annual national study of teacher supply and demand has emphasized these four facts:

1. In the elementary schools, where the need for competent teachers is greatest, the supply of newly qualified candidates has increased encouragingly, and as many as four of every five of these new eligibles actually enter classroom service.
2. The supply of candidates for high school teaching decreased each year for five years, and about one of every four of these new eligibles entered occupations other than teaching.
3. For five years the supply of newly qualified candidates for the teaching of mathematics and the sciences decreased even more rapidly, and one of every three of these did not enter teaching.
4. The increasing shortage of high school teachers of mathematics and science has been paralleled by an expanding shortage of such teachers in colleges and universities.

As these changing conditions became apparent, the teaching profession—both as individuals and organized groups—began some years ago to intensify the effort to identify young students with aptitude in scientific fields and to encourage the continued study of mathematics and the sciences in particular. Specifically, teachers everywhere have been quietly at work for some years seeking the ends now most loudly demanded by the alarmists. The result has been a steady strengthening *of the proportion of all college graduates who prepare for teaching*, along with the *greatest percentage increase* in the number preparing to teach mathematics and science.[1] The accomplishments are to the credit of educational workers without the spur of Russian fireworks or the help of the would-be saviors of the public schools.

Despite these encouraging achievements, however, the situation remains critical. Sufficient funds have not been provided to induce many of these newly qualified eligibles to enter teaching. Even more tragic is the accelerated rate of loss of good teachers from classroom service. As will be shown in a later paragraph, the demand for replacements, particularly in mathematics, has zoomed since 1952.

Recent supply statistics

It is not easy to survey in advance the supply of persons available for any professional service. First, the choice of occupation is always an individual matter; choice of a program of college preparation for a specific occupation does not require actual entrance into it. Second, new entrants into the teaching of mathematics may come from any of these sources:

1. The most recent class of college graduates with major preparation in mathematics.
2. The most recent class of college graduates with major preparation in some other subject and minor preparation in mathematics.
3. Graduates of earlier years, both those who prepared for teaching but have not yet entered classroom service and those who have prepared themselves for teaching through postgraduate study.

[1] Up this year, 18.8 per cent in mathematics and 18.4 per cent in science, compared with a general increase of 10.5 per cent in all high school teaching fields combined.

4. Former teachers who return.
5. Others—generally those with only transitory interest in teaching and those employed despite their obvious lack of needed qualifications.

Members of the first group can be identified prior to graduation. Thus the Research Division report issued in March each year indicates the number in this new "supply" who will be available for employment the following September. Those in the other four groups are not "new," inasmuch as they have been in the general population and could have entered teaching at an earlier date if they had chosen to do so. The fact that they did not so choose casts doubt upon the likelihood that they will turn to teaching in large numbers unless the attractions (salaries) should become more competitive. It is for this reason that only the members of the first group—members of the current graduating class with preparation for teaching—are reported as the new "supply."

Table 1 shows the record since 1950. That year is chosen because it saw the greatest production of college graduates for all occupations and also because the complete figures were not available earlier. Column 3 of the table shows a steady drop through 1955 and only a partial recovery since. To make these figures meaningful, however, it is necessary to answer this question: How many of this newly produced supply will enter teaching? Of the 3,068 qualified graduates of 1957, only 68.1 per cent, or about 2,100, were found to be in teaching service last September.

TABLE 1

COLLEGE GRADUATES WITH MAJOR PREPARATION TO TEACH MATHEMATICS

YEAR	NUMBER OF GRADUATES	PER CENT CHANGE FROM 1950
1	2	3
1950	4618	—
1951	4118	−10.8
1952	3142	−32.0
1953	2573	−44.3
1954	2223	−51.9
1955	2155	−53.3
1956	2544	−44.9
1957	3068	−33.6
1958	3633	−21.3

TABLE 2

ESTIMATED NUMBER OF NEW MATHEMATICS TEACHERS EMPLOYED

YEAR	NUMBER EMPLOYED TO TEACH		TOTAL
	Full-time mathematics	More than half-time mathematics	
1	2	3	4
1952	1638	1485	3123
1953	2112	1676	3788
1954	2300	1618	3918
1955	2723	1750	4473
1956	3044	1905	4949
1957	3920	2558	6478

It has not yet become possible to identify and classify all new teachers, but the 1958 survey was complete in thirty states. Considering them to be representative of nationwide conditions, and extending the sample to all schools, the estimated number of newly employed mathematics teachers is shown in Table 2.

Certainly the shortage of competent candidates for mathematics teaching is critical, but the figures in Tables 1 and 2 must be interpreted with severe limitations. At first glance it would appear that because only about 2,100 of the mathematics majors graduating in 1957 entered teaching while almost 6,500 new mathematics teachers were employed, the resulting difference of 4,400 new teachers were all unqualified. *Such an assumption would be grossly inaccurate.*

Research techniques have not yet been developed for the analysis of the qualifications of all new teachers, but some, perhaps a good many, of these 4,400 new teachers came from:

1. The most recent graduating class with strong *minor* preparation in mathematics.

2. College graduates who since graduation have made satisfactory preparation for teaching.
3. Fully prepared graduates of earlier years now returning from military service or now turning from other occupations, graduate study, or homemaking.
4. Former teachers with successful records.

Obviously, none of these can be identified in the general population until they decide to apply for teaching positions, and thus they are not classified in the "supply." But it is common knowledge that many of the newly employed teachers enter with meager or totally inadequate qualifications. Only further research can reveal their exact number.

Turnover is alarming

On the surface, Table 2 would seem to indicate a great expansion of mathematics offerings in the high schools since 1952. In that year (the first for which the data are available) an estimated 3,100 new teachers entered service. Five years later, in 1957, almost 6,500 new mathematics teachers were employed.[2] This does not mean, however, that the total staff of mathematics teachers was enlarged.

On the contrary, these figures chiefly reflect the steady annual increase in the number of mathematics teachers withdrawing from the classroom service. The figures reflect the rapid upsurge in the demand for such persons in other occupations. Even more to the point, the figures reflect the fact that many school districts are unable to retain good teachers of proved ability in the face of better employment opportunities outside teaching. These are the sobering facts which merit the attention of the general public. Critics of the schools must be convinced that demands for improvement must be matched by the resources necessary to attract and hold superior teachers in classroom service.

[2] Throughout this report, the teacher who changes from one position to another is completely ignored because he neither fills nor creates a vacancy; he only transfers the demand for a teacher from one location to another. A "new" teacher is one who did not teach anywhere the previous year.

Meeting of national interest

The 58th Annual Convention of the Central Association of Science and Mathematics Teachers, built around the general theme, "The Challenge of Science and Mathematics in a Free World," will be held on November 27–29, Claypool Hotel, Indianapolis, Indiana.

Dr. H. J. Muller, Nobel Prize Winner, and a number of other well-known scientists and mathematicians will appear on the program. Tours to Pitman-Moore Company, Allison Division of General Motors, and Instruction Center, Indianapolis Public Schools, have been arranged. The Physical Science Study Committee, Cambridge, Massachusetts, and the UICSM, Urbana, Illinois, will each give a 3½-hour program. Sectional programs followed by a luncheon and two speakers from the U. S. Office of Education will terminate the program.

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Creative teaching in mathematics

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Three levels of teaching mathematics are explored, and it is asserted that the product of creative teaching is insight.

PROVIDING EXPERIENCE in which the child is led to discover, to create, to figure out largely by his own efforts many ideas, principles, or relationships rather than to be told them, is coming to have first place in teaching, not only in mathematics but in many other areas as well. The word best suited to describe this way of directing learning has not yet evolved. Some have recently called it "Developmental Teaching"; more recently and perhaps more widely in mathematical literature it has been designated as "Discovery Teaching." In art and music and in the elementary school a very similar idea thrives under the category of "Creative Teaching." To create implies bringing something into existence and to discover seems to indicate the more or less accidental uncovering of something previously existing. Therefore, since for the child the idea is new and created in part by him, even though it has always existed, we shall adopt the phrase "creative teaching" and use this paper to describe and to illustrate it.

In order to get a perspective on procedures commonly used, we might recognize that in general there are three distinct levels of teaching, or, if you count a zero level, one might speak of four levels. The zero type of operation completely misses the audience, and few if any students are able to learn anything from it. A German lecturing in his native tongue to an American audience would illustrate this zero level of teaching. Similarly, a mathematics teacher using the language of mathematics, or a teacher in any area who uses a technical vocabulary before it is mastered, may sometimes be speaking in a "foreign" language. If a teacher uses terms and concepts whose mastery has not been previously established, then his work will often be on this zero level. In an Extension class a teacher told of a child's response to the question, "How many feet are in a yard?" A bright child said, "Miss Jones, I don't see how you can tell how many feet there are in a yard unless you know how many children are playing there." A high school teacher reports that one student spelled slide rule, "sly drool." A student in a geometry class, in a test dealing with sides of a triangle, spelled sides, "s-i-z-e," so it was evident that he failed to follow the thinking and that the teacher had been performing on that zero level for him.

In addition to this "null" learning, there are three common, distinct, and reputable types or levels of teaching in mathematics. They are briefly described as: (1) teaching the rule; (2) telling the student everything, explaining every step for him to understand, to remember, and to follow; and (3) creative teaching. They are distinct not only in procedure but also in philosophical foundation and in their psychological implications and consequences. Although distinct, they are frequently used in combination. Illustrations of these types may help to keep this paper above the "null" area to which reference has already been made.

RULE TEACHING

Many years ago a group of students at Columbia University visited one of the New York high schools. The teacher took roll, then began her teaching about as follows:

"You have just learned how to add and how to subtract signed numbers. Today we will learn how to multiply them. The rule is easy. All that you need to remember is that like quantities give plus and that unlike quantities give minus.

"I will write some numbers on the board:

$$5 \times 4 = 20$$

$$(+5) \times (+4) = +20$$

"What is the rule, George?"

GEORGE: "Like quantities give plus and unlike quantities give minus."

TEACHER: "Good. $+5$ and $+4$ are both plus, therefore they are like quantities, and the answer is $+20$. Now if I change the signs of both numbers, they will still be like quantities. Therefore, $(-5) \times (-4)$ also equals $+20$. Suppose I change the sign of only one of the numbers, then they will have unlike signs, and the last part of the rule applies. That is, $(-5) \times (+4) = -20$ or $(+5) \times (-4) = -20$. The reason is that unlike quantities give minus.

"Now turn to page 53 and we shall work some of those problems. Mary, will you try the first one?"

MARY: "$(+8) \times (+5) = +40$."

T.: "Why is the answer $+40$, and not -40, Joe?"

JOE: "Like numbers give plus."

T: "Excellent! Andrew, the next one."

ANDREW: "$(-8) \times (+9) = -72$."

T.: "Why is the answer -72, and not $+72$, Susan?"

SUSAN: "Unlike numbers give minus!"

T.: "Very good! Sue, take number 3."

SUSAN: "$(-8) \times (-6) = +48$, because likes give plus."

FRANK: . . . etc. . . .

T.: "Work this whole set of 50 problems for tomorrow. Again, what are the rules, Kate?"

KATE: "Likes give plus and unlikes give minus."

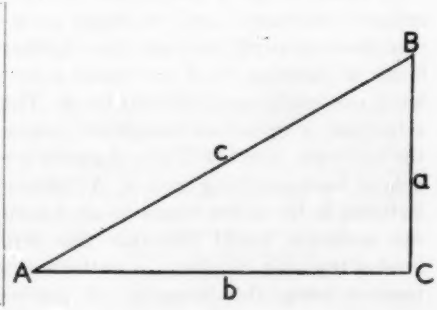
T.: "Good! The rules again, Sam!"

It will be needless to point out the limitations of this teaching. Even the rules were poor. The sum or difference of 2 quantities with like signs could be + or −, and the product of 3 quantities with like signs could be + or −. Good rule-teaching would have made the rule: "The product or quotient of two quantities with like signs is positive, and the product or quotient of two quantities with unlike signs is negative."

A similar experience in learning trigonometry in college is indelibly impressed upon my memory. The first day began about as follows:

TEACHER: "Trigonometry involves six new words which I shall write on the board in a column and in such a way as to help you remember them. Their abbreviations and definitions are as follows. They refer to the right triangle adjacent.

sine	$\sin A = \dfrac{\text{side opposite}}{\text{hypotenuse}} = \dfrac{a}{c}$
cosine	$\cos A = \dfrac{\text{side adjacent}}{\text{hypotenuse}} = \dfrac{b}{c}$
tangent	$\tan A = \dfrac{\text{side opposite}}{\text{side adjacent}} = \dfrac{a}{b}$
cotangent	$\cot A = \dfrac{\text{side adjacent}}{\text{side opposite}} = \dfrac{b}{a}$
secant	$\sec A = \dfrac{\text{hypotenuse}}{\text{side adjacent}} = \dfrac{c}{b}$
cosecant	$\csc A = \dfrac{\text{hypotenuse}}{\text{side opposite}} = \dfrac{c}{a}$



"Notice that the last three are the reciprocals of the first three, as shown by the vertical bars on the left. If you learn them in this order they will be easy to remember, because the first letter is also the last letter, the second is also the second from the last, etc. The letters form a symmetrical pattern.

"Trigonometry involves remembering these definitions and the formulas and equations that come from them. For example:

$$\tan A = \frac{a}{b}, \text{ then } \frac{a}{b} = \tan A \text{ and}$$

$$a = b \tan A.$$

"Notice that to multiply tan A by b one always writes b at the left, b tan A. Also note that tan $B = b/a$ or $b = a$ tan B. These definitions you must learn and remember. You should be able to apply them to any triangle, no matter how lettered, and to use them in the form of the original definitions or in the form cleared of fractions."

Enough said. Another trigonometry lesson will be discussed later.

Telling everything: a lecture

A second commonly used method assumes that the student is more than a machine that is to be trained to perform certain mechanical tricks. Consequently, this level of teaching is far above the rule level. It assumes that understanding is necessary, and one who teaches on this psychological plateau insists that understanding must precede rule learning. He therefore explains and illustrates every step, and the students follow faithfully, and finally learn the rule. There is satisfaction in the fact that there seems to be a logical reason for everything, and students acquire a deeper appreciation for and a more functional mastery of the techniques or principles to be learned. This is fortunately the most common kind of good teaching. One illustration should help to make it clear. Perhaps it will also make clear that while great emphasis is put on understanding, yet little premium is placed upon developing leadership, resourcefulness, or self-confidence.

Mrs. Jones is a geometry teacher at Western Hills High School and a great believer in visual aids to help children understand geometry. Today she will demonstrate the formula for the area of a circle. Previously she has established that $C = \pi d$ or $C = 2\pi r$, and that π is a constant and is a little more than 3.

Mrs. Jones: "I have cut up a circle into 36 sectors, 10° each, and have mounted the pieces with half of them pointing up and the other half fitted in but pointing down. Notice that the figure looks like a parallelogram, except for its wavy upper and lower bases. The area of a parallelogram is its base times its height. The base is half the circumference and the height is the radius. Now we shall substitute these facts in the first formula and see what follows." She writes on the board:

$A = b \cdot h$

$= 1/2C \cdot r$, since $b = 1/2\ C$ and $h = r$.

$= \pi r \cdot r$, since $C = 2\pi r$, $1/2C = \pi r$.

$= \pi r^2$. The formula for area of a circle is $A = \pi r^2$.

"Now let's go back to the wavy base. Suppose that I had cut this circle into 72 sectors, or 360, or perhaps a million sectors. The figure would then be more like a rectangle and the base more nearly straight. However, the four steps we took to develop the formula would not be changed. The height would still be the radius; and the base, half the circumference. Therefore, the area of a circle is expressed by the formula $A = \pi r^2$."

This lesson was actually taught by using a part-discovery approach after the teacher showed the cut-up circle, as follows. The presentation was changed a bit to illustrate a lecture lesson. Such a combination of lecture and creative teaching is not uncommon.

Teacher: "What does this circle now look like?"

MARY: "It looks like a parallelogram, but its bases are not straight lines."

T.: "Very good! Omitting consideration of its wavy bases, what would be its area?"

GEORGE: "If it's a parallelogram, its area is its base times its height."

T.: "What is the height in terms of the circle?"

GEORGE: "The height is the radius."

T.: "Now let's see if you can figure out the length of the base in terms of the circle."

SUSAN: "The lower base is half the circumference, since the other half forms the upper base."

T.: "Excellent! Now let's collect what we have and see what comes of it." She writes on the board:

$$A = bh.$$

T.: "Now what shall we substitute for b and h from the circle?"

GEORGE: "$h = r$ and $b = \frac{1}{2}C$."

T. (as she writes): "Then, $A = \frac{1}{2}Cr$. But what is the formula for C that we got recently, Ann?"

ANN: "$C = \pi d$ or $2\pi r$."

T.: "Then what is $\frac{1}{2}C$, Ann?" etc., etc., etc.

CREATIVE OR DISCOVERY TEACHING

As an illustration of creative teaching, or discovery teaching, we choose a lesson in a trigonometry unit for a geometry class taught by Mr. Adams.

TEACHER: "When would the sun be at an angle of 45° above the horizon?"

GEORGE: "In the morning at about 9 o'clock."

MARY: "Also in the afternoon at about 3 o'clock."

T.: "Very well. Suppose that we were out in the school yard when the sun's rays hit the ground at an angle of 45°. If we measured the length of the shadow of the flagpole, and found it to be 36 ft., would we know anything about the height of the pole?"

GEORGE: "We'd know the height of the flag pole because the triangle would be isosceles."

T.: "Suppose the shadow of a tree at the same time is 42 ft., how high is the tree?"

MARY: "The tree would have to be 42 ft. high, because it too forms with its shadow a 45° right isosceles triangle."

T.: "What would you say would be true about the ratio between the side opposite the 45° angle and the side adjacent to it in any 45° right triangle?"

HARRY: "Those sides would be equal, so the ratio would have to be 1 to 1 or just 1."

T.: "Very well. Now listen carefully. If this ratio of the side opposite the angle to the side next to the angle is always equal to 1 for a 45° right triangle, regardless of the size of the triangle, what would it be for a right triangle with an angle of 35°?"

GEORGE: "I think the ratio would be less than 1, because the side opposite would always be less than the side adjacent."

MARY: "I think we could draw such a triangle and measure the sides to find the ratio."

T.: "Do you think the ratio would be the same for any triangle, so long as it is a right triangle with a 35° angle in it, regardless of its size?"

GEORGE: "If they were all right triangles with a 35° angle, the triangles would all be similar and the ratios between corresponding sides would be equal."

T.: "Henry and Anne, will you go to the board and each construct as carefully as you can a 35° right triangle. Henry, will you make your base 30 inches, and Anne, make your base 10 inches long."

He experimented thus for a 35° triangle, then with a 30° triangle, to have them feel that the ratio depended upon the angle alone, not on the size of the triangle. Then he concluded about as follows:

TEACHER: "You have been playing with one of the most important concepts in mathematics, the ratio between the side opposite an acute angle and the side ad-

jacent to the acute angle, and have discovered that the ratio depends upon the angle only. The Arabs discovered this principle over 1000 years ago and for many centuries this ratio has been called the *tangent of the angle*. That is, in any right triangle, the ratio between the perpendicular sides, the side opposite divided by the side adjacent to an angle, is called the *tangent of the angle*, abbreviated 'tan.' What then is the tangent of 45°?"

GEORGE: "Tangent seems just a short name for the ratio we have been talking about, so the tangent of 45° = 1."

T.: "Very good, George. We computed the tangent of 35° and 30°. What was our approximate value for these ?"

MARY: "We found tangent 35° = about .7 and the tangent 30° a little less than .6."

T.: "In your book, p. 247, you will find a table of tangents which have been computed by accurate methods. Find there the tangent of 35° and of 30°."

SAM: "Tan 35° = .7002, tan 30° = .5774."

T.: "Notice that in the table the tangent of 60° = 1.7321. Tell me now what that means to you."

JIM: "That must mean that the ratio between those perpendicular sides is 1.7321, or that the side opposite the 60° angle is 1.7+ times the shorter side adjacent."

T.: "Very good, Jim. Now let's see how a surveyor could use this ratio and this accurate table to solve problems."

Enough, perhaps, to reveal the teacher attitude in discovery teaching. By means of a realistic setting and skillful questioning, basic relationships, fundamental procedures, and usually the final rule or generalization often can be developed largely by the students. Definitions must be told by the teacher; they cannot be discovered. For example, in this lesson the definition of tangent was told, but the principle that the ratio is a function of the angle alone was discovered or created by the children through careful planning and guidance by the teacher.

Another example:

Mr. Naugle believes in discovery teaching and he also believes that mathematics serves its highest potentialities when it is useful as well as rigorous. The subject of cutting speeds had come up in the machine shop and the boys had brought it into the geometry class. This involves the definition of new technical terms and the derivation of a formula to determine the relation between the revolutions per minute (rpm) of the drill, its diameter in inches, and the cutting speed in feet per minute.

TEACHER: "Does anyone have a mathematics problem this week to share with the class, especially one that involves the geometry of circles which we are now studying?"

GEORGE: "In machine-shop work today we needed to know the rpm for a half-inch drill to give it a cutting speed optimum for cast iron, 100–120 feet per minute."

T.: "That is an excellent problem, George, and one which the whole class might like to help solve. Before they can think clearly on it, however, we must have a definition of rpm and cutting speed because some may not know these terms and may think that cutting speed means the rate at which the drill penetrates the metal. George, what are rpm and cutting speed as the machinist defines and uses these terms?"

GEORGE: "Cutting speed is peripheral speed, or the rate in feet per minute that the outer tip of the cutting edge contacts the material. The same definition applies to grinding and polishing wheels. There, cutting speed is the speed in feet per minute at which the wheel contacts the material. Rpm means revolutions per minute."

T.: "To find the cutting speed, that is, the number of feet which the outer tip of the cutting edge travels in a minute, how can we begin?"

SAM: "I'd suggest that we find first how far it moves in one revolution of the drill, then we can multiply that by the rpm to find the total distance traveled in a minute."

T.: "That sounds reasonable, Sam. That is excellent thinking. Can someone follow Sam's lead now and tell us how to find the distance the tip travels in one revolution?"

MARY (among several volunteers): "That would be the circumference of the drill."

T.: "Suppose now we agree on some letters to use for the formula. Let us use s to represent the desired cutting speed, r for the rpm, and c for circumference. What would be our beginning formula?"

GEORGE: "$s=rc$, but since $c=\pi d$ we can write it $s=r\pi d$ or $s=\pi rd$."

T.: "This is a good formula except for the dimensions used: s has to be in feet per minute and d is always given in inches. The drill used was a one-half inch drill. How can we change the right-hand side to feet?" (Comment: Perhaps the children could have been led to discover this aspect of the formula, but this teacher told them.)

MARY:

$$“s=\frac{\pi rd}{12}.”$$

T.: "In machine shops the rpm is not very precise. For example, 2000 rpm's may be inaccurate by even a hundred rpm's. Therefore, for this purpose a machinist simplifies this formula by using $\pi=3$. How would this change the formula?"

SUSAN:

$$“s=\frac{\pi rd}{12}=\frac{rd}{4}.”$$

T.: "Very good, now in this formula d will be used in inches, yet S will be in feet, and the constant $\pi/12$ or $\frac{1}{4}$ accounts for the change in units. This is the machinist's formula for cutting speed,

$$s=\frac{rd}{4}, \text{ or } 4s=rd.$$

Can you find the cutting speed of a drill revolving at 1000 rpm for a $\frac{1}{2}$-inch drill?"

GEORGE:

$$“s=\frac{rd}{4}=\frac{1000\cdot\frac{1}{2}}{4}=125 \text{ ft./min.}”$$

T.: "Now let's go back to George's original problem. He wanted to know the rpm for a given cutting speed and diameter. Can this formula be solved for rpm?"

SAM:

$$“\text{If } s=\frac{rd}{4}, \text{ then } r=\frac{4s}{d}$$

and George's problem becomes

$$r=\frac{4\cdot 100}{\frac{1}{2}}=800.$$

George would have to run his drill at about 800 rpm to have a cutting speed of 100 ft. per minute."

T.: "Will someone now restate the formula, tell us what it means and how it is used."

This was well done and the machinist's job completed. The incomplete dimensional analysis can be overlooked. Perhaps this teacher had an unusually bright class to come up with such good answers. When he first began to teach this way he no doubt had to ask more questions and expect smaller discoveries. Now students know what to expect and are amazingly alert. They love it, and resent having the teacher tell them portions which they can discover.

We summarize our three levels of teaching: by rules, by meaningful lecture, by discovery or insight. A great teacher once taught that now abideth hope, faith, and charity, these three, but the greatest of these is charity. So we would paraphrase that: now abideth skill, understanding, and insight, these three, but the greatest of these is insight, the product of creative teaching.

Edited by Howard Eves, University of Maine, Orono, Maine

A history of computers, II[1]

by Jules A. Larrivee, Mathematician, Lockheed Aircraft Corporation, Burbank, California

HIGH-SPEED ELECTRONIC DIGITAL COMPUTERS

The idea of using telephone relays as components of an automatic digital computer was first introduced by G. R. Stibitz, and a computer built along these lines was demonstrated in 1940. Since that time improvements have been made and several such computers built by Bell Telephone Laboratories are in daily use.

The Second World War created an emergency which made it imperative to develop much faster methods of computing. Lives could be saved by finding rapid and accurate solutions to pressing military problems. During the war years, J. P. Eckert, Jr. and J. W. Mauchly, then at the Moore School of Electrical Engineering of the University of Pennsylvania, designed and built an automatic high-speed digital computer. It was called the ENIAC (Electronic Numerical Integrator and Calculator). Its basic components were vacuum tubes, some 18,000 of them being used. The use of vacuum tubes in such a computing machine was a new idea and not too much was known about the possible behavior of tubes under the exacting conditions of performance imposed on them. The failure of tubes and other components was one of the major problems that had to be overcome.

The increase of speed of operation over other computers was one of the distinguishing features of the ENIAC. The time for an operation such as an addition or a subtraction was measured in milliseconds and the entire operation was automatic. The ENIAC was made known to the public in 1946, and since that time, governments, industry, and universities have built electronic digital computers.

Among the high-speed digital computers available for purchase or rental may be listed the following: Types 650, 701, 704, 705, and 709 of the International Business Machines Corporation; the Univac, the Univac Scientific, and the File Computer of Remington Rand; model E 101 of the Burroughs Corporation; the Datatron of the Electro-Data Corporation; and the Elecom 125 of the Underwood Corporation. These vary in speed, cost, and storage capacity. A listing of computers available to business (up to 1955) will be found in an article by J. M. Carroll in *Electronics* of June 1955. The Stretch Computer by IBM and the Larc Computer by Remington Rand are now in developmental stages and will have speeds of operation 100 times faster than those available today. In addition, they will have much more storage.

Many universities have built electronic computers. Among these we mention the Illiac at the University of Illinois, the computer at the Institute for Advanced Study, the APEXC at Birbeck College of the University of London, the Whirlwind at the Massachusetts Institute of Technology, and the several computers built at Harvard University. Pioneer work in the development of electronic high-speed computers was done at the University of Manchester in England. Digital computers have been built in England, Germany, Sweden, Holland, Italy, France, and the

[1] Continued from the October 1958 issue.

Soviet Union. Originally, these projects were sponsored by universities or government agencies. The latest information regarding computer work, both in domestic and foreign fields, will be found in the "Digital News Letter" published in the *Journal of the Association for Computing Machinery.*

The National Bureau of Standards has built two computers: SWAC (Standards Western Automatic Computer) and SEAC (Standards Eastern Automatic Computer). The Naval Research Laboratory built the NAREC.

Basic parts of an automatic computer

The basic parts of an automatic computer may be listed as input, arithmetic unit, storage (memory), control, and output. Computers are either serial or parallel. In a serial computer the pulses representing the digits travel along one conductor, the time difference in the several pulses serving to distinguish between different digits and different combinations of digits. In a parallel computer there is an individual conductor for each digit. For this reason parallel computers are faster but require more equipment.

The input to automatic computers has taken various forms, such as punched cards, punched paper tape, and magnetic tape. Perhaps the most convenient way in which to get information into the input medium is by means of a keyboard. Most high-speed computers operate internally in the binary system, and the data must be changed into this form before insertion into the arithmetic unit. The binary system has only the two digits 0 and 1, and an electronic component which can take one of two possible states (on or off) is ideally suited to represent digits in the binary system. Further, addition and multiplication are much simplified, and the resulting circuits are also simplified. Subtraction is performed by adding complements and division is repeated subtraction. Thus the basic operations are addition and shifting.

The arithmetic unit is that part of the computer where the actual arithmetical operations are carried out. Most of the large-scale parallel binary computers made use of Eccles-Jordan flip-flop trigger circuits as basic components in their arithmetic units. These trigger circuits had been known since 1919, but their widest use has probably been in computer construction. Transistors have been replacing tubes as components; they are smaller in size, and since they generate less heat they reduce the need for cooling devices. Magnetic amplifiers have also been used as replacements for tubes in some computers, one such having been built by Sperry-Rand for the Air Force Cambridge Research Center.

Punched cards, relays, and stepping switches represented early forms of storage in the development of storage or memory media. Some of the requirements for a storage medium for a high-speed digital computer are that it must have a rapid access time (the time necessary for getting information out of the storage once it has been put in), that it should be relatively permanent (although this does not preclude the necessity for regenerating it from time to time), and that it should be of sufficient capacity to store data in an amount which would normally be expected to occur in the types of problems to be solved on the computer. These various requirements were found to be satisfied to a higher or lower degree in several media—mercury delay lines, electrostatic tubes, magnetic drums, magnetic tape, and magnetic cores.

The mercury delay line, or acoustical delay line, operates by having an information pattern inserted into a path containing a delay element. By the use of amplifiers and timing circuits, this information is fed back into the circuit, making a closed loop. Electrical impulses are changed into sound waves by a transducer; these sound waves then travel down the mercury column and are changed back into electrical impulses. Originally the BINAC, the Univac, and the SEAC all made use of mercury delay lines.

The electrostatic tube is used as a storage medium; charges representing the information are stored on the face of a cathode-ray tube. The charges have to be regenerated at frequent and regular intervals. The Williams Cathode Ray Tube is perhaps the most widely used of these and found application in the IBM Type 701, the SWAC, and the Illiac.

Magnetic drums are used as intermediate storage in many computers. The practical realization of such drums was pioneered by Engineering Research Associates. If sufficiently high speeds are possible, the magnetic drum is an important adjunct to other more rapid access types of storage. Information is stored in the form of magnetized spots on the surface of the drum. Writing and reading heads are provided, by means of which information is put on and taken off. One important feature of the drum is that the record is not destroyed by power failure and there is no problem of regeneration.

Magnetic tape is also used as storage. The time needed to get information out is longer than for some other types of storage because often the whole tape must be searched to find a particular item. However, it is compact and is valuable in "off-line" printing.

In the forefront at the present time, as working storage, is the magnetic core memory. The recognition of the possibility of using magnetic cores with rectangular hysteresis loops as components of storage was due to J. W. Forrester and J. A. Rajchman. Such storages have rapid access times, a necessary property if they are to be used with the high speeds now available in arithmetic units. The size of the individual core is about that of the letter "o" in the printed word *core*.

Other techniques that have been used for storage media are photographic film, magnetic cells, ferroelectric cells, capacitor-diode memories, and many others.

The control circuits of the computer determine the order in which the various operations are performed. They also insure that one operation is completed before the succeeding one is initiated. Coded instructions to the computer, through the intervention of the control, will determine when a transfer is to be made and when the computer will stop, enter, or leave an iteration procedure.

The output from a computer can be put on punched cards or tape, but what is usually wanted is a sheet of paper with the information printed on it in terms of our usual decimal notation. Because of the high speed of operation of the computer, the printing devices are often not able to print the information as rapidly as it comes from the computer. Some printing is done "off-line" from cards or tape; the use of high-speed printers makes it possible to do some of it "on-line."

High-speed printers which work directly off a computer are usually either of the wheel-and-hammer type or of the wire type. Synchroprinter, developed by Anelex of Boston, is an example of the wheel-and-hammer type; Burroughs has a wire type designated as ser G. Electronic high-speed printers have been developed that print 1000 lines a minute.

Using a cathode-ray tube called the charactron, Convair Division of General Dynamics Corporation has developed a fast printing device with a rate of about 100,000 characters a second. Similar to the charactron is the typotron, developed by the Hughes Aircraft Corporation.

Several so-called dry printing processes make one or more copies of a cathode-ray screen or other device. Haloid of Rochester, New York, has a process called xerography which performs such a function.

Mention should also be made of smoke printing, under development at Standard Register in Dayton, Ohio. It is an electrostatic process making use of a fine pigmented mist or smoke to develop a latent electrostatic image.

Other developments

A computer which combines some features of both analog and digital types is known as the Digital Differential Analyzer (DDA). It is basically a counting

computer, operating by counting electrical pulses. However, data can be input in either digital or analog form and the output can be of either type. Because of its moderate cost and this dual capability, the DDA will probably be used to a greater extent as time goes on.

That information on analog and digital computers might need to be compared was early recognized and much work has been done on design of analog-to-digital and digital-to-analog converters. In one type of the latter, digital values cause shaft rotations which are proportional to voltages, these being the analog representations of the digital quantities. A complete system consisting of a high-speed digital computer and a large analog computer, the two connected by converters, is under development at Ramo-Wooldridge. Thus the speed and accuracy of the digital computer will be augmented by the continuous recording of data of the analog computer, and the processes of data sampling and simulation can be studied more thoroughly.

Conclusion

Many problems in science, engineering, and business whose solutions once seemed to be beyond reach are now being successfully attacked by the use of high-speed electronic digital computers. Their use has made possible the construction of more complex mathematical models, with a resultant expansion and increase of our knowledge in many fields. In the business world, problems which demand an enormous number of elementary operations, such as inventory and payroll calculation, now are done in a small fraction of the time previously required. The automatic digital computer is also becoming the central element in control systems that are the basis of our progress toward automation.

Analog computers have demonstrated their usefulness in the continuous recording of data from wind-tunnel tests and similar problems. Their low cost and ease of construction are important considerations in making a choice as to the type of computer to be used.

The nonspecialist may have little or no direct contact with computers of any kind. Nevertheless, he should regard himself as a part-owner of such machines, because his Government, through purchase or rental, has invested large sums of money in them.

Pappus's extension of the Pythagorean Theorem

by Howard Eves, University of Maine, Orono, Maine

Every student of high school geometry sooner or later becomes familiar with the famous Pythagorean Theorem, which states that *in a right triangle the area of the square described on the hypotenuse is equal to the sum of the areas of the squares described on the two legs.* This theorem appears as Proposition 47 in Book I of Euclid's *Elements,* written about 300 B.C.

Even in Euclid's time, certain generalizations of the Pythagorean Theorem were known. For example, Proposition 31 of Book VI of the *Elements* states: *In a right triangle the area of a figure described on the hypotenuse is equal to the sum of the areas of similar figures similarly described on the two legs.* This generalization merely replaces the three squares on the three sides of the

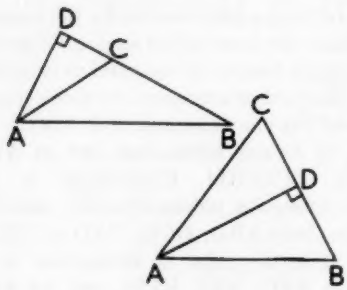


Figure 1

right triangle by any three similar and similarly described figures. A more worthy generalization stems from Propositions 12 and 13 of Book II. A combined and somewhat modernized statement of these two propositions is: *In a triangle, the square of the side opposite an obtuse (acute) angle is equal to the sum of the squares on the other two sides increased (decreased) by twice the product of one of these sides and the projection of the other side on it.* That is, in the notation of Figure 1, $(AB)^2=(BC)^2+(CA)^2\pm 2(BC)(DC)$, the plus or minus sign being taken according as angle C of triangle ABC is obtuse or acute.

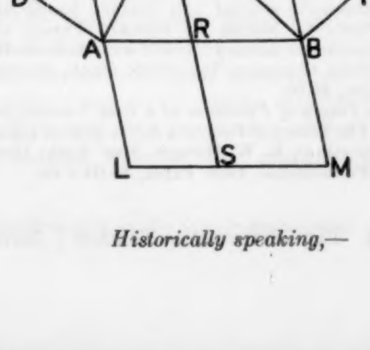
If we employ directed line segments we may combine Propositions 12 and 13 of Book II and Proposition 47 of Book I into the single statement: *If in triangle* ABC, D *is the foot of the altitude on side* BC, *then* $(AB)^2=(BC)^2+(CA)^2-2(BC)(DC)$. Since $DC=CA\cos BCA$, we recognize this last statement as essentially the so-called *law of cosines*, and the law of cosines is indeed a fine generalization of the Pythagorean Theorem.

But perhaps the most remarkable extension of the Pythagorean Theorem that dates back to the days of Greek antiquity is that given by Pappus of Alexandria at the start of Book IV of his *Mathematical Collection*. Pappus, the last of the great Greek mathematicians, flourished toward the end of the third century A.D., and with high enthusiasm and marked competence tried to rekindle fresh interest in the languishing Greek mathematics. Although Pappus wrote a number of commentaries on important Greek works of mathematics, his really great contribution is his *Mathematical Collection*, a combined commentary and guidebook to the existing geometrical works of his time. This work of Pappus is sown with numerous original propositions, improvements, extensions, and historical remarks, and it has proven to be a veritable mine of rich geometrical nuggets.

The Pappus extension of the Pythagorean Theorem is as follows (see Fig. 2): *Let* ABC *be* any *triangle and* CADE, CBFG any *parallelograms described externally on sides* CA *and* CB. *Let* DE *and* FG *meet in* H *and draw* AL *and* BM *equal and parallel to* HC. *Then the area of parallelogram* ABML *is equal to the sum of the areas of parallelograms* CADE *and* CBFG. The proof is easy, for we have $CADE=CAUH=SLAR$ and $CBFG=CBVH=SMBR$. Hence $CADE+CBFG=SLAR+SMBR=ABML$. It is to be noted that the Pythagorean Theorem has been generalized in two directions, for the right triangle of the Pythagorean Theorem has been replaced by *any* triangle, and the squares on the legs of the right triangle have been replaced by *any* parallelograms.

The student of high school geometry can hardly fail to be interested in the

Figure 2



Pappus extension of the Pythagorean Theorem, and the proof of the extension can serve as a nice exercise for the student. Perhaps the more gifted student of geometry might like to try his hand at establishing the further extension (to three spaces) of the Pappus extension: *Let* ABCD (see Fig. 3) *be any tetrahedron and let* ABD-EFG, BCD-HIJ, CAD-KLM *be any three triangular prisms described externally on the faces* ABD, BCD, CAD *of* ABCD. *Let* Q *be the point of intersection of the planes* EFG, HIJ, KLM, *and let* ABC-NOP *be the triangular prism whose edges* AN, BO, CP *are translates of the vector* QD. *Then the volume of* ABC-NOP *is equal to the sum of the volumes of* ABD-EFG, BCD-HIJ, CAD-KLM. A proof analogous to the one given above for the Pappus extension can be supplied.

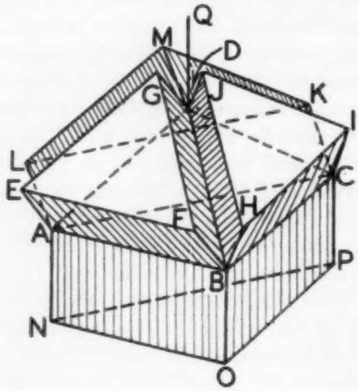


Figure 3

What's new?

BOOKS

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A Short Course in Differential Equations (2nd ed.), Earl D. Rainville. New York: The Macmillan Company, 1958. Cloth, x+259 pp., $4.50.

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The Theory of Functions of a Real Variable and *The Theory of Fourier's Series* (Dover republication), E. W. Hobson. New York: Dover Publications, 1958. Paper, $6.00 a set.

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Elements of Mathematical Biology (Dover republication), Alfred J. Lotka. New York: Dover Publications, Inc., 1956. Paper, xxx+465 pp., $2.45.

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Figurets, J. A. H. Hunter. New York: Oxford University Press, 1958. Cloth, x+116 pp., $3.50.

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Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

"3 x 5" Review

by Dwight W. Allen, Hillsdale High School, San Mateo, California

It was near the end of the semester. I was reviewing in first-year algebra class by putting multiple-choice questions from old examinations on the board in outline form. I deliberately selected some poorly constructed questions as a part of the review. Here is an example:

One factor of a^2-a-6 is:

a. $(a+3)$	*d.* $(a-5)$
b. $(a-3)$	*e.* $(x-3)$
c. $(a-4)$	*f.* none of the above

As I discussed each question, such as this one, with the class, I asked them to tell me why the incorrect choices, the foils, would perhaps be logical if the wrong method were used. While some of the choices were obviously wrong (as most are in the case of the above example) and the incorrect alternatives were immediately identified by various members of the class, others took considerably more reasoning. I asked the class to suggest better foils for poor questions. We would correctly solve the problem on the board. (The factors of a^2-a-6 are $(a-3)(a+2)$.) After looking at the correct solution, the students agreed that $(a+3)$ was a good foil choice. Someone suggested that if a student confused his signs he might think $(a-2)$ was a correct factor. Another student said that a student who was looking for a monomial factor might think that "a" by itself was a factor. A girl (who had made the mistake herself on several occasions) said we ought to include the choice that "there are no factors; it is not factorable." We agreed that "none" was always a good choice—and should sometimes be the correct choice. The question, as we finally agreed upon it, was in this form:

One factor of a^2-a-6 is:

a. $(a-3)$	*d.* a
b. $(a+3)$	*e.* the expression is not factorable
c. $(a-2)$	*f.* none of the above

The questions clearly provided the kind of refresher activity needed at the time, providing the learning activity and holding the attention and interest of the pupils for a full class hour. This, in turn, laid the groundwork for an assignment from which I was to obtain the questions for my final examination in algebra and also to obtain a wide variety of material for additional review.

Our work during the first year had consisted of seven or eight major topics with subdivisions under each. I told the students that the best way for them to prepare for the final examination was to ask themselves, "What is important—what would I ask questions about if I were making out a test on this material?" I then made the following assignment: Each student was to prepare 30 questions suitable for use on a final examination. Each question was to be written separately on a 3×5 card with the answer on the back. Questions were of the multiple-choice

variety that were suitable for answering on a standard answer sheet with from four to six choices. On the back of the card, in addition to the answer, they were to write their reasons for asking the question and sign their names.

The questions would be evaluated in this way: One third on coverage: how carefully were topics selected, and how adequately did the questions cover the topics? One third on test construction: how effective were the foils they suggested as alternative answers, and how clearly did they demonstrate awareness of other pitfalls they should avoid? The remaining third was based on the reasons they gave for including each question. They were told to leave out any purely trick questions, and to be sure that the reasons they gave were more substantial than "it is important to know. . . . "

In discussing the assignment and the reasons for it, it was agreed that, if the quality of the questions was good—and there was every reason for supposing it would be—then I would agree to select the questions for their final examination entirely from the questions they submitted. After grading the questions, all those that were suitable for use would be returned to the class for review.

The assignment was made on a Wednesday and was due the following Monday. The over-all quality of the questions I received was excellent. Here is an example of one of the difficult questions I selected to use:

Write the following expression using only positive exponents:

$$\frac{x^{-2}}{-2y^{-1}}$$

a. $\dfrac{-x^2}{2y}$ *d.* $\dfrac{x^2}{-2y}$

b. $\dfrac{2y}{-x^2}$ *e.* $-\dfrac{x}{2y^2}$

c. $\dfrac{2y}{x^2}$ *f.* none of the above

The correct answer,

$$-\frac{y}{2x^2},$$

was not one of the alternatives, and the correct choice was (*f*).

These questions represented considerable time spent by the pupils in reviewing the material to select the topics of their questions and then to construct good questions. Pupils not only became conscious of scope of the material they were to cover, but had to think specifically through typical difficulties in order to formulate the alternate foils for the multiple-choice answers. They agreed afterward that this assignment resulted in a much more purposeful and comprehensive review than they would have otherwise made.

I told them that it was perfectly legitimate for them to show their cards to their friends, to discuss them, and even to change them so long as the basic effort was their own. Thus encouraged, they spent time in comparing questions and obtaining much constructive criticism—and as a result more review and a better quality of work turned in to me.

After collecting the question cards and appraising them on the three criteria outlined above, many were obviously inappropriate because of being too easy, too difficult, covering obscure material, poorly constructed, etc. In all, about one-third to one-half of the questions were discarded for these reasons. Fifty questions were finally selected for a 45-minute test. In some cases they were edited to standardize the form, or to revise some of the alternate answers. However, a minimum of changes was necessary, and the questions represented in the main the work of the pupils. Even less editing would have been necessary had I not desired to include, insofar as was possible, one question from each student. This was important, for it was a definite mark of success for a failing student to see one of his questions on the final exam—and, needless to say, one question did not change the

outcome. After the examination was written, the questions used were returned to the group of those generally acceptable.

The next day a review session was scheduled using the question cards, thoroughly shuffled, for two algebra classes. Each student was given three question cards with an additional supply placed in the front and in the back of the room. Every student was allowed to study each question to his satisfaction, check his answers by looking on the back, determine the source of his error, if any, and pass the card on in a prescribed order. If anyone ran out of questions before others were passed to him, he could obtain more from either the front or the back of the room. In case of doubt about a question or an answer, the pupil could raise his hand and I would help him. When a general problem appeared, I would discuss it with the class, using the board. It would be erroneous to give the impression that all of the questions were difficult. A good share of them covered very basic and simple material—as was proper for a general review of the course. An example of such a question would be:

Perform the indicated operation: $x+x^2$

a. $2x^2$	*d.* $2x^3$
b. $2x$	*e.* $x+x^3$
c. x^3	*f.* none of the above

Pupil reaction to this procedure was extremely favorable. They enjoyed the review, and because of the novelty, spent more time on it than they otherwise might have. Knowledge that the actual test questions were among the cards passed around during the general review made the whole process even more exciting. From my point of view this was justifiable, because any student who could know the answers to the approximately 1000 review questions would know enough algebra to meet the highest course requirements!

The best questions—the very top quality—initiated a file which is to be built up over the years as a source of test items. Obviously, it was not necessary to limit this practice to the final examination at the end of the year. A similar review pattern could be used at any time during the course. Such an item file would solve permanently the problem of giving make-up tests. From a student-constructed file of several hundred questions covering each topic considered during the year, 50 questions could be pulled out at any time for a make-up test over any given part of the course. This general procedure is adaptable to review and testing in other fields in the high school—in areas of skill, information, and understanding. Perhaps other teachers will find such additional uses for this "3×5" review.

The following item is a selection from the world's first encyclopedia, a compendium of what men knew or believed, produced by a Spanish bishop, Isidore of Seville, in the seventh century.

Some even numbers are excessive, others are defective, others perfect. Excessive are those whose factors being added together exceed its total, as for instance, XII. Defective numbers are those which being reckoned by their factors make a less total as, for example, X. The perfect number is that which is equalled by its factors, as VI (2 plus 3 plus 1). The perfect numbers are VI, XXVIII, CCCCXCVI.—*Taken from* A History of Education *by Luella Cole.*

• NEW IDEAS FOR THE CLASSROOM

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

Variety has been called the spice of life. It is also the spice that stimulates learning. Most of us would like to add variety to our lessons, but we frequently run dry of ideas. It is the intent of this new section to report a variety of new ideas that mathematics teachers far and wide have found successful. To do this we need to have material supplied by you. Send in your unique idea to the editor so that others may profit from your success. Send your idea whether it be great or small. And send us information about people who are trying out new ideas so that we can ask them to share their creativeness in this section. The success of the section is dependent on the contributions of its readers. Send us your ideas today.

Enriching instruction via television

by Emil Berger, Monroe High School, St. Paul, Minnesota

A series of mathematics programs on the local educational television channel was a successful avenue for presenting enrichment topics to St. Paul and Minneapolis schools. These half-hour programs, presented at 1:30 every Wednesday during the months of January, February, and March, reached a wide audience of secondary school students. The programs were in the nature of lessons designed for ninth- and tenth-grade mathematics classes. Descriptive previews of the lessons were circulated before the opening of the series. The descriptions are reproduced below. The descriptions suggest that the lessons highlighted exciting and important concepts. The objective of the series was to stimulate interest in mathematics and to teach many important mathematical concepts usually not included in textbooks or courses. The series was also planned as an experimental study to find out how much pupils learn from viewing a mathematics lesson on a TV screen.

In order to evaluate the learnings of the viewers, tests were given prior to selected lessons and immediately following them. The same tests were also given to control groups who did not view the lessons. At the end of the series a questionnaire was given to both pupils and teachers viewing the lessons to get their subjective reactions.

For the instructor it was an exceedingly wearisome task to plan each lesson, to design and build props, and to practice and time each lesson. As a basis for practice and also to compare learnings by TV with the classroom presentation, each lesson was presented by the TV instructor to his tenth-grade mathematics classes. These classes took the same tests as the experimental and control groups mentioned above.

Descriptions of the lessons

1. *How precise is that measurement?* The initial lesson on enrichment topics was planned around problems involving noth-

ing more sophisticated than reading and recording the measurements of lengths and perimeters of certain objects. Different scales which are in common use were presented in the settings in which they are used. In this subtle way the foundation was laid for giving meaning to such concepts as precision of a measurement, accuracy, relative error, and per cent of error. Finally, consideration was given to rules for performing computations with numbers that represent measurements.

2. How large is that surface?
The meaning of the concept of area was developed with the aid of space-filling objects such as marbles, regular hexagons, and unit squares. The process of finding areas of plane figures was introduced as an operation of counting unit squares. Formulas for finding the areas of triangles, rectangles, and parallelograms were developed intuitively with the aid of concrete referents. Formulas were then used as computational aids for finding areas of figures whose dimensions are numbers resulting from measurements. Computation of areas was used as a basis for discussing the topic of significant digits.

3. Slide rule I
Instruction in the use of the slide rule was conducted for two consecutive sessions. In the first lesson the slide rule was introduced as a mechanical device for performing computations. A large demonstration rule was employed to make viewing easier. Instruction was limited to learning necessary vocabulary, reading the C and D scales, and finding products and quotients. In preparation for beginning work with the slide rule prospective viewers were encouraged to supply themselves with inexpensive rules which have A, B, C, and D scales; however, this was only in the nature of a suggestion.

4. Slide rule II
In the second lesson on the slide rule a brief review of the work of the previous lesson was used as an introduction. The main part of the lesson dealt with an explanation of the mathematics underlying the construction of the slide rule. As preparation for this lesson students should have review work involving the definition and laws of exponents.

5. All about π
π was introduced as the ratio of the circumference to the diameter of a circle by means of some clever, concrete illustrations. Following a discussion of the significance of the ratio definition, a brief résumé of the origin and adoption of π as the symbol for the ratio in question was presented. The main part of the lesson dealt with an account of some of the classical efforts that have been made to assign a value to the ratio C/d. In conclusion, several interesting applications of π were described.

6. Indirect measurement
By way of introduction to the topic of indirect measurement some simple but interesting illustrations of indirect measurements that anyone can complete with his own hands, feet, and eyes were described. The difference between direct and indirect measurements was carefully explained. The major part of the lesson dealt with explanations of how to use devices like pocket mirrors, fish poles, fruit jar covers, and paper mailing tubes to find inaccessible distances and heights.

7. Map making
This lesson was in the nature of a demonstration of some simple methods of making maps of small areas such as backyards, lakes, and irregular patches of land. Though the equipment employed was homemade, the equipment used by the professional surveyor in completing the same kinds of tasks was displayed and described.

8. Map projections
The problem of representing the surface of the earth on a flat sheet of paper was explained with the aid of appropriate demonstration materials. Equal area and

conformal projections were displayed and compared. The difference between the Mercator and tangent cylinder projections was carefully illustrated.

9. Geometrical constructions without straightedge and compasses
The usual geometrical constructions were effected with devices such as the T-square, carpenter's square, and parallel rules, and the techniques of paper-folding. Attention was also given to methods of drawing the ellipse, parabola, hyperbola, and cycloid.

10. Pythagorean relation
The lesson was introduced with a brief history of the use of the 3-4-5 triangle relation before the time of Pythagoras. An account was presented of how Pythagoras probably discovered the proof of the theorem that bears his name. Finally, attention was given to various dissection puzzles and applications.

11. Probability
Mathematical probability was discussed in connection with coin tossing, games of chance, and drawings. Empirical probability was discussed in connection with quality control and life expectancy.

12. Binary number system
Several easily understood schemes for writing numbers from the decimal scale in binary notation were presented in detail. Computations were performed with numbers written in the binary scale. Computing machines were introduced as an application of the binary scale.

A mathematics exhibit

by Vivian Strand, Burlington College, Purlington, Iowa

For several years my mathematics students have been asked to complete a project of their own choice. These projects were completed outside of class and could be anything mathematical that was also useful, beautiful, or intriguing. Each project was due on a definite date, at which time the student would demonstrate his completed project to the class. In this way the assignment became an experience in creativeness, craftsmanship, and communication. It is such experiences that our students need if they are to grow in appreciation and understanding of the elegance and power of mathematics.

Since performance in mathematics does not get the publicity surrounding that of athletics, music, and the drama, it was decided that an open house be held in the mathematics department. This would afford an opportunity to exhibit projects and to present demonstrations.

The guests were greeted by "George Geometry," a life-sized robot created by two of the students from paint cans, downspouting, and other scrap material. Two students conducted soap-film experiments by forming soap bubbles, using frames in the forms of cubes, tetrahedrons, and circles. They displayed a curve-stitching model of a hyperbolic paraboloid and showed the same figure in their soap bubbles. They also demonstrated harmonic motion. The visitors challenged a mechanical brain (assembled from a kit) to a game of ticktacktoe. The machine also worked logic problems, and at the same time a student showed the principles involved by means of blackboard drawings.

The game room was very popular. Younger visitors liked the three-dimensional ticktacktoe and struggled over puzzles such as one consisting of odd-shaped pieces of wood which, when prop-

erly assembled, formed a cube. Adults were challenged by a variety of problems. All were intrigued by a mathematical kaleidoscope made of buttons, screws, rubber bands, and other odds and ends whose reflections produced beautiful designs, no two alike. Two students demonstrated the Moebius strip and other interesting glimpses of topology. They had people guessing what would happen if one piece of paper with three half-twists was cut in half and another piece of paper with two full twists was divided into thirds. Examples of practical problems encountered on his job were explained by another student.

The properties of the sine curve and the cycloid were demonstrated by means of models. The conoidal or "sea shell" roof and the geodesic dome were the most unusual exhibits of modern architecture. Notebooks on advertising, chemistry crystals, and photography as well as mobiles added variety and atmosphere to the exhibit.

The response of the visitors as well as of participating students was one of keen interest. And now that the class has returned to the normal routine of solving problems and writing tests, they are already planning ahead for a bigger and better exhibit next year. For inspiring favorable publicity and stimulating projects an exhibit is well worth the effort.

Christmas in the mathematics classroom

by Donovan A. Johnson, University of Minnesota, High School, Minneapolis, Minnesota

For several years I have kept a folder in my file labeled "Christmas." It is now filled with materials that have been used for pre-Christmas activities in many mathematics classes. Appropriate activities during the week before Christmas and especially the last period before vacation can leave both the teacher and the pupil with a better attitude toward mathematics. And your students will return to work when vacation days are over with an expression that is like $y=x^2$ rather than $y=-x^2$!

One of the most attractive projects is a Christmas tree decorated with geometric solids. Suggestions for this project will be found in THE MATHEMATICS TEACHER for December 1948 and May 1953. Constructing these brightly-colored plane and solid geometric figures can be a learning activity as well as an opportunity for creativity. Unusually unique and attractive decorations can be made by forming stellated polyhedrons. These are made by attaching pyramids to the faces of the regular polyhedrons. A variety of simpler designs can be made by paper-folding, using construction paper or gift wrapping paper.

A somewhat similar project is the construction of mathematical greeting cards. These usually are made of colored geometric designs and greetings that involve mathematical ideas. The hexaflexagon has many possibilities. Formulas or graphs may convey ideas connected with the holidays. Some teachers give the students an assignment in which the answers spell out an appropriate greeting. This can be done by solving formulas, reducing algebraic fractions, substituting lettering for numbers, or using the first letter of each answer to spell out a greeting.

Another project is the preparation of a

bulletin-board display. A suitable display is one on snowflakes. The geometric shapes, the amount of moisture in a snowfall, the number of snowflakes that fall, the probability of snowflakes being alike, and statistics on snowfalls are all appropriate for discussions and problems. And do not forget a quotation about Christmas.

A variety of problems can be related to the season. Try some code problems like this addition problem

```
 FINE
 TOYS
-----
TODAY
```

or this multiplication problem

```
 CUT
   A
----.
TREE
```

Other puzzles like these can be adapted to the Christmas season. GREETINGS! Try these on the family and your friends during Christmas vacation.

1. Johnny and Mary went Christmas shopping with their folks. When they went to the toy department a big Indian and a little Indian rode up on the elevator with them. The little Indian was the big Indian's son, but the big Indian wasn't the little Indian's father. Can you explain it?
2. When they got to the toy department Johnny's father met a friend he had not seen in years. With his friend was a little girl. "I'm glad to see you," said the friend. "Since I last saw you I've been married—to someone you never knew. This is my little girl."

   "I'm happy to meet you," said Johnny's dad to the little girl. "What's your name?"

   "It's the same as my mother's," she answered.

   "Oh, then your name is Anne," said Johnny's dad. How did he know?
3. When Johnny's dad took out his wallet to pay for the toys he had 6 bills totaling $63, yet none were $1 bills. What were the 6 bills?
4. While they were waiting for the clerk to make change, Johnny said to Mary: "I have in my pocket two coins which add up to 55¢ but one of them is not a nickel. What are they?"
5. At the gift-wrapping counter they watched three busy girls wrap 3 gifts in 3 minutes. How long would it have taken 100 girls to wrap 100 presents?

Some teachers plan a mathematical Christmas party. For this the invitations, food, decorations, and games all revolve around mathematical ideas. Mathematical games, tournaments, tricks, and puzzles are used for entertainment. Even gifts are exchanged, but these "gifts" are only descriptions of gifts. In this event, the descriptions are couched in mathematical terms even though the gift described is nonmathematical, such as a pair of skis or a football. The Christmas season would be a good time to talk about the mathematics of toys. The Magic Designer, Mecinette, Erector Sets, Tinker Toys, or puzzles are based on mathematical ideas as well as being usable to contruct mathematical models. The ratios of lengths, the size of angles, and the relationships of lengths in linkages can involve excellent applications of mathematics.

Even the game Scramble, although not related to Christmas, might be suitable vocabulary practice for the day before Christmas.

Scramble

The object of this game is to match a drawing, picture, or article with a familiar mathematical term. Pictures to illustrate these terms can be found in magazines. Mount the pictures on cardboard and display them around the room. Players move from picture to picture to identify the terms. The person identifying the most terms in a given time is the winner. To "assist" the players, the mathematical terms to be used are supplied, albeit in a scrambled spelling and in scrambled order.

Sample items for Scramble

Picture or exhibit	*Mathematical term*	*Scrambled spelling*
1. vegetables, such as carrots, radishes, beets	roots	tosor
2. sunburned farmer	tangent	gnanett
3. coffee being poured from a coffeepot	hypotenuse	synthepoeu
4. advertising sign	sine	nies
5. electric lines	power	wroep
6. identical twins	similar figures	ramliis sufegir
7. political extremist	radical	cirlada
8. Stalin	locus	soluc
9. houseboat	arc	car
10. oleomargarine	substitute	bistttuues
11. cash or down-payment sign	terms	sremt
12. king or queen or dictator	ruler	rerlu
13. ax or hatchets	axes	sexa
14. airplane	plane	lepna
15. fat lady	round number	nudor merbun
16. empty bird cage	polygon	yonoplg
17. prison	prism	simrp

Christmas formula:

$$SG=\int_{'58}^{'59}[MC+HNY]dt$$

when SG = Seasons' Greetings

MC = Merry Christmas

HNY = Happy New Year.

Letter to the editor

Dear Sir:

I have read several issues of THE MATHEMATICS TEACHER and found them extremely interesting. Because my work is concerned with digital computer applications, the articles describing the use of these computers and the teaching of programming for them are of particular interest.

I am happy to see that mathematics and computing are being introduced to students earlier in their scholastic careers than has been the case in the past. As proof that students can absorb this knowledge at a younger age than one might suspect, I submit the enclosed method for computation of logarithms devised by my eight-year-old son Stephen after a brief introduction to logarithms and extensive study of a table of logarithms. The proof was furnished by one of my associates, Raymond Kramer of the Ramo-Wooldridge Corporation, who was anxious to verify the adequacy of the method.

Very truly yours,
Robert A. Beach
5624 Calle de Ricardo
Torrance, California

To compute $L=\log_B N$ to d fractional digits, first compute:

$$Q=N^{(B^d)}$$

in the number system radix B.

Then let S equal one less than the number of digits to the left of the radix point in Q when $N\geq 1$, or the negative of one more than the number of leading zeros in the fraction Q when $N<1$.

Finally, $L=S\cdot B^{-d}$; i.e., place the radix point d places to the left of the rightmost digit in S.

Proof: From the definition of S it follows that

$$B^S\leq N^{(B^d)}<B^{S+1},$$

and since $B^L=N$ from the definition of logarithms,

$$B^S\leq (B^L)^{(B^d)}<B^{S+1}$$

$$B^S\leq B^{L\cdot B^d}<B^{S+1}$$

$$S\leq L\cdot B^d<S+1$$

$$S/B^d\leq L<S/B^d+1/B^d$$

$$0\leq L-S/B^d<1/B^d.$$

Therefore S/B^d approximates L to within $1/B^d$, or to d fractional digits.—*Stephen C. Beach.*

A column of unofficial comment

College mathematics in the high school

by D. M. Merriell, University of California, Santa Barbara, California

At the beginning of the semester, one of my analytic geometry students stood out from the rest of the class. When I talked with him after class one day, I learned that he had taken a course in analytic geometry at his small-city high school. By the end of the semester, his performance had become undistinguished and he failed my final examination.

Another student in the same class came from a large-city high school. His I.Q. was unusually high and he had taken a course in mathematical analysis offered to a select group of students. In my first test, he made B. Thereafter he failed every test but squeezed through the course. This situation continued in the introductory calculus course the next semester. As a result, a boy of above-average intelligence who had intended to major in mathematics dropped mathematics completely.

A third student came from a good suburban high school. There he had taken analytic geometry. He was bright but not inclined to systematic work. His test grades alternated between B and F and he, too, failed the final examination. In the second semester his work was never above D. His stated intention of becoming a physics major was almost certainly doomed to failure.

Are these cases atypical? Conversations with other college mathematics teachers lead me to think otherwise. I believe we are witnessing a phenomenon that was not anticipated by those who favored moving segments of college mathematics into the high schools. Curriculum committees should give this careful thought.

In a certain city, a high school committee outlined its objectives in introducing a course called Mathematical Analysis. The one-semester course was to consist mainly of analytic geometry, taught from a standard college text, plus a very brief introduction to the ideas of calculus. One of the objectives was to introduce the students to some of the concepts which would be developed in college mathematics; another, to provide stimulation for students who expected to continue with mathematics in college. A representative of the committee stated in a public discussion that part of the motivation for the course was the feeling that the transition to college mathematics was difficult and such a course would make the student's initial college experience less painful.

The three cases I have cited show the other side of the coin. In the end, the introduction of these students to college life was more, not less, painful. They were under the illusion that they were in for an easy time and when experience showed them to possess no particular advantage over their fellows, their disillusion prevented them from bringing their natural ability to the rescue. When they began to slip, there was no challenge left to tide them over.

The objective of stimulating able high school students is not only praiseworthy, it is of prime importance. The introduction of college mathematics courses for

this purpose has the sympathy and backing of almost all college teachers. It should be to the advantage both of the high school student and of his teacher. But to use college material for dubious social or psychological aims is not the use which the colleges have in mind. Furthermore, it introduces the danger, so frequently observed by national review commissions, of needless repetition of material in school and college.

The time is certainly ripe for the introduction of year-long analytic geometry-calculus courses of college standard in the high schools. Many private schools and some high schools already give such courses, and the Advanced Placement Program of the College Entrance Examination Board offers a means for evaluating the results. Most colleges will not hesitate long over accepting these courses for advanced placement, if not for credit, once their caliber has been established.

If a high school does not feel itself ready to offer such a course but still wishes to provide additional challenge for its best students, what should it do? I suggest that a semester of analytic geometry is *not* a good solution even if it can be offered at college standard. The objection lies in the growing popularity of the combined calculus-analytic geometry course. A student who enters a college which follows only this program will be no better off for having had a semester of analytic geometry. Indeed, he will be forced to suffer through spells of needless repetition, with possible adverse effects.

What alternatives are there to analytic geometry? Several paths are still open to the ambitious high school teacher. One is to offer a course based on the program of the University of Illinois Committee on Secondary School Mathematics, or on the University of Chicago general mathematics course. Another alternative is to select topics from "finite mathematics." And another is to give a course in Theory of Equations, a subject which seems regrettably to be on the way to extinction in the colleges. In fact, the latter course would fulfill almost all of the objectives laid down by the city high school committee without doing the disservice of introducing the student to material which he will have to learn again.

In summary, I believe high school curriculum committees which wish to introduce college mathematics should postpone courses in analytic geometry and calculus until they can meet the standards of the Advanced Placement Program. In the meantime, there are excellent alternative courses available to make the high school teacher's work more interesting and to challenge the superior student.

The curriculum of present-day mathematics is set forth largely in terms of 19th-century technology, whereas 20th-century mathematical methods have played a part in transforming physical theory from classical Newtonian bases to modern quantum mechanics bases. Furthermore, new mathematics methods in statistics and the theory of decision-making have pervaded the biological and social sciences to such an extent that the introduction of these elements into the education of the citizen at an early age seems desirable.—*Alan T. Waterman, Director, National Science Foundation, as reported in* Engineering and Scientific Manpower Newsletter.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Basic Mathematics, H. S. Kaltenborn, Samuel A. Anderson, and Helen H. Kaltenborn (New York: The Ronald Press Company, 1958). Cloth, ix+392 pp., $4.75.

This text is designed to bridge the gap between simple arithmetic and the traditional freshman courses in college algebra, trigonometry, and analytic geometry. Therefore, it rests on the former, introduces the latter, and spans the subject matter ordinarily covered by elementary and intermediate algebra. The last two chapters present an excellent introduction to statistical processes, probability, and sampling.

It may be used for a terminal course in general education with emphasis on the basic skills of algebra and/or an introduction to trigonometry and analytics. Or it may well serve as a guide for review, self-study, or classroom instruction for those students desiring to continue in the traditional sequence.

The authors have tried to present basic concepts and understanding along with basic skills, as far as it is possible at this level. Naturally there is no attempt to start with a minimum set of postulates. Rather, the student is presented with basic rules for executing the operations of arithmetic and algebra. In the first chapter, the positive or negative sign is used as a superscript before the number to distinguish the sign of the number from the operational symbols for addition and subtraction. This is an excellent idea, though it seems to be discontinued too soon and with too little notice.

One looks for mechanical solutions. The section on solving verbal problems presents some nine suggestions, but these are quite logical in nature. There is no attempt to set up a work page or form that is supposed to solve the problem automatically without any active analysis on the part of the student. However, in some instances the presentation would be too mechanical for some classroom teachers, as in the solution of quadratic equations by use of the formula. Only once is the solution by completion of the square employed, and that in the development of the formula. In fairness one should mention that the formal solution was preceded by the graphical solution, which may well be the best basis for understanding the quadratic equation.

Again one looks for sources of misconception that have too often been present in elementary texts. The authors are careful to state that the radical of order "n" is used to refer only to one of "n" roots, and that there are certain limitations in simplifying the product or quotient of two radicals. In the solving of equations the student is cautioned to be aware of the loss of a root or acquisition of an extraneous one.

The composition and printing are good, though conservative by some standards. The basic rules and developed theorems and formulas are not printed in bold type or colored ink. The figures, plates, and tables are well arranged and attractive, yet not spectacular. The examples and exercises are abundant in number and have been chosen with care. The answers to odd-numbered exercises are printed in the appendix, and the others may be obtained from the publishers.

Actually one has to teach from a book for a semester or two before giving a true evaluation of it. A new book is like a new friend—only time will tell the story. In the case of this book, first impressions are good. As stated in the preface, *Basic Mathematics* offers considerable flexibility in teaching, according to the student's competence in mathematics and the purpose of the particular course.—*R. G. Smith, Kansas State Teachers College, Pittsburg, Kansas.*

College Mathematics (College Outline Series), Kaj L. Nielsen (New York: Barnes & Noble, Inc., 1958). Paper, xviii+302 pp., $1.95.

This book is not intended to be used as a textbook, but the author claims it "is complete in itself" and "should provide any reader with a thorough knowledge of the usual topics studied during the first year in this field." This reviewer questions the words "complete" and "thorough" in this statement. The book is keyed to twenty textbooks, yet too many theorems are given without proof and the explanations for several are poorly done.

For example, the explanation on page 9 of the laws of exponents for negative and fractional exponents is not at all clear. Also on page 9 this is found in two illustrations: $\sqrt{16}=\pm 4$ and $\sqrt{4}=2$. The portrayal of synthetic division on page 157 might well have been omitted, for it lends no light on the quotient and very little on the remainder.

Many chances to tie things together with at least a reference were missed. For example, the distributive and associate laws are given on page 6, whereas on page 7 rules for using parentheses and brackets are given as seemingly additional laws to be obeyed. Although the term "identity"

had been defined earlier, it was not used in the proof of the Remainder Theorem. This, however, is not the only fault found in this proof, the last steps of which are (page 156):

$$f(r)=(r-r)Q(r)+R.$$

Since $Q(x)$ is a polynomial in x, $Q(r)$ is a number and:

$$(r-r)Q(r)=0 \qquad Q(r)=0$$
$$\therefore R=f(r).$$

Working formulas for the general cubic equation are given (page 165) with no mention of the irreducible case. Moreover, the one example used has a rational root. The

$$\lim_{v\to 0}\frac{\sin v}{v}=1$$

is given (page 218) with no mention of radian measure.

The treatment of calculus is brief, not covering many topics now considered first-year college mathematics in many places. The first two pages on calculus (pp. 212–213) will confuse any reader because, in finding a limit of $\Delta y/\Delta x$, both Δx and x are made to change.

The following are further examples of "bad spots" (the underlining is the reviewer's):

p. 23 "the average rate of change of y in Illustration 1 at each point"

p. 39 "steadily increases in size as θ varies from 0 to $\pi/2$ and finally becomes larger than any number we have."

p. 69 "the two important parts are: part I, the verification for $n=1, 2, 3, \cdots k$; and part II, the proof that if it is true for $n=k$, then it is true for $n=k+1$."

p. 152 "Illustration 2. In how many ways can a hand of thirteen cards be dealt from a deck of fifty-two cards so as to contain exactly ten spades, if the spades are selected first?"

p. 213 "$S=\dfrac{a}{1-r}=\dfrac{1}{1-\frac{1}{2}}=2$. Thus the limit of S is 2."

I will find this book handy because of its three pages of references, by topics, to pages of twenty texts. As a basis for study, however, one might better choose one of the list of twenty.—*Kenneth W. Wegner, Carleton College, Northfield, Minnesota.*

Elements of Plane Trigonometry, Henry Sharp, Jr. (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958). Cloth, ix+274 pp., \$4.95.

The primary distinctive feature of this new textbook is the admirable attempt to offer trigonometry in the terminology and spirit of *modern* mathematics. Prefaced by a brief introduction to the theory of sets, the book consistently utilizes the language of sets. Emphasis is placed not on degree measure but rather on radian measure. The book contains a rather extended and generally excellent treatment of both the general theory of periodicity and its special role in trigonometry. The topic of trigonometric equations and identities is handled in a relatively careful manner. Brief reference is made to Fourier analysis. Methods of coordinate geometry are employed both in defining the trigonometric functions and in deriving basic trigonometric relationships (such as the law of sines). Frequent use is made of the method of proof by mathematical induction. In general, terms are carefully defined.

Yet the selection of content here will probably not satisfy completely either traditionalists or moderns. Obviously, traditionalists will be unhappy with the modern reduction of emphasis on solution of triangles and logarithmic computation. They may regret the omission of such topics as the law of tangents, the half-angle tangent formulas, co-logarithms, and an introduction to spherical trigonometry. Many moderns will be disappointed at the omission of discussion of such topics as the definitions of the trigonometric functions in terms of exponentials and/or infinite series; trigonometric functions of a complex variable; the hyperbolic functions; applications of vector methods in developing and applying the theory of the trigonometric functions (less than one page is given to the topic of vectors); and the use of trigonometry in developing the modern geometry of the triangle and circle. Some teachers will be unhappy because little or no attention is given to such topics as polar coordinates; the exponential *function* (the base "e" is not even mentioned); special limits such as $\lim_{x\to 0}\frac{\sin x}{x}$; or the existence of versed sines (and coversed sines, haversines, exsecants, etc.).

Still other teachers may object to the treatment of such material as the applications and history of trigonometry. In general, applications of trigonometry to fields such as surveying and science are postponed until the ninth chapter. Little or no attention is given to the theory of computing instruments (such as the slide rule) and measuring instruments (such as the transit), applications of logarithms to fields such as business and psychology, or applications of trigonometry to topics such as trajectories in artillery and refraction of light. Little *detail* is provided for the applications of periodic aspects of trigonometry in the theory of music and alternating currents. Teachers who prefer strong "doses" of history in mathematics will *not* find here such names as Vieta, Napier, Briggs, Euler, DeMoivre, or Cauchy. However, brief, interesting glimpses into the ancient work of Ptolemy are emphasized. Certainly, the allotment of sixty pages or so (about one-fourth of the book) to logarithmic tables is regrettable in the light of both the author's *modern* intentions and the possibility of using much of this space for extending included topics or presenting a bibliography or other topics.

In general, the readability of the book is good, although some explanations seem much too brief. Italics are effectively used, but some of the printing is occasionally blurred. The tables and figures are generally satisfactory, although they frequently could have been improved by the insertion of titles. Answers are provided for most odd-numbered exercises (to be assigned to students), but the exercises are neither classified as to difficulty nor generally arranged in order of difficulty. There is some use of such effective pedagogical devices as "informal" language and illustrations of theorems *prior* to proof. However, too often definitions and techniques are "handed" to the reader rather than "discovered" by the reader by inductive questioning or laboratory procedures.

While the presentation is generally relatively careful, some drawbacks consist of minor errors, misprints, inconsistencies, and awkward notation or phrasing. The definition of a prime number (p. 17) needs clarification as to the nature of the factors (negative, fractional, irrational, etc.) not permitted. The statement on the central angle (p. 20) contains the unnecessary words "the intersection of." Reference is made (p. 158) to the *amplitude* of a complex number, without previous definition or explanation. (The *argument* of a complex number is defined on p. 157.)

Furthermore, one finds such items as "isoceles" (p. 101) and "system" (p. 20); the statement "$\sin \pi(0/r)=0$" (p. 37), rather than "$\sin \pi = 0/r = 0$"; and the notation "$\pi/4(45°)$" (p. 26), rather than "45° or $\pi/4$." On page 122, it is stated that a capital letter (such as B) will be used to denote a vertex of a triangle and a corresponding Greek letter (such as β) will be used to denote a vertex angle of the triangle, but, on page 124, one reads "$B=90°$." On the same page, one finds the product of two trigonometric functions indicated both with and without the dot notation. (See, for example, pages 91, 94, and 99.) The dot is used, too, between two parentheses (pages 98 and 160) and between a natural number and a surd (p. 103).

On page 112, the reader may get the impression that historically the symbol "arc sin x" was devised to avoid the possible misinterpretation of the symbol "$\sin^{-1} x$." (No mention is made of the separate historical evolution of the symbols in England and on the European continent as a partial outgrowth of the conflict over the "discovery" of the calculus.) Answers (on p. 259) to exercises (on p. 16) use the system of identifying quadrants by Roman numerals before the text itself indicates (p. 17) that such a system is to be used.

The book is one to be considered for possible use in modern courses in college or secondary-school trigonometry, but it would not be appropriate for use with relatively weak students. While the book is far from perfect, it is a noteworthy step in the right direction. Both author and publisher deserve commendation for courage in pioneering in the field of modern trigonometry.—*Herman Rosenberg, New York University, New York, New York.*

Fantasia Mathematica, edited by Clifton Fadiman (New York: Simon and Schuster, 1958). Cloth, xix+298 pp., $4.95.

Logic seen through the lens of fantasy becomes *Alice in Wonderland;* religion and humor can be combined in *The Screwtape Letters.* One wonders what a mixture of mathematics and imagination would produce. Clifton Fadiman gives the answer in these stories and poems that he has collected with evident glee. Let us not cavil and complain that often the results show little mathematics, for they are all fun; all we must be assured of is that a quick-reading public will not infer the inverse—that if it is mathematics it is not fun. The general impression for one who reads through the whole book, as I did, on a cross-country plane, is of having eaten too much fudge: too much of the same, good diet. Taken in small bits, the selections are clever, refreshing, and full of mathematical twists startling to those of us who sometimes take our work too seriously.

The poems and short quotations are too elusive to classify, but contain many favorites convenient to have in a safe place. Although the stories are almost as difficult to classify, let us try it: four are based on the idea of the Moebius strip; two each on the following—arithmetic, the Pythagorean Theorem, combinations and permutations, and electronic computers; and one on each of these—Fermat's last theorem, the tesseract, the fourth dimension, the four-color problem, compound interest, rapid accelerations, and the task of teaching mathematics in college. In addition there is one short, slightly suggestive mathematical joke, and a longer selection from *Jurgen* that is either mathematical mysticism or clever salaciousness, depending upon your background and honesty.

One does get tired of the properties of the Moebius strip, but in general the over-all correctness of the mathematics and the originality of its application evoke your admiration. Since the purpose is only to entertain, it seems pedantic to note that most selections can be used to interest students in the serious mathematics behind the façade, or to encourage them to search out similar mathematical curiosities in order to construct similar fictions. It may be enough to relax and be amused.—*Henry W. Syer, Kent School, Kent, Connecticut.*

Intermediate Algebra for Colleges, Gordon Fuller (Princeton, New Jersey: D. Van Nostrand Company, Inc., 1958). Cloth, vii + 258 pp., $3.90.

This text, like many of the type designated as "Intermediate Algebra," is intended for the college student who appears with less than the necessary minimum of knowledge to succeed in the regular college algebra course. The first three chapters (covering Positive and Negative Numbers, Literal Numbers and the Fundamental Operations, and Introduction to Equations) are especially written with a view to helping the student who has forgotten this part of his high

school algebra. These chapters, if taken slowly, should provide the student with the foundation for the material that follows the regular outline of essential topics through quadratic equations. The material is inclusive enough that the text could be used for a terminal course as well as a preparation for those who will pursue the subject further, or who may take up the mathematics of finance and elementary statistics.

The format of the text is traditional; no attempt has been made to "open up" the pages of the print, or to "lighten up" the graphical illustrations. However, a compensating feature of the format is the wide use of italics and boldface type in drawing attention to rules, definitions, and directions. "Warnings" and "cautions" to the students are not to be found either, and this we think is to the book's credit, for they often misteach more than they teach.

The discussion and explanation of the text follow the pattern of Dr. Fuller's other three earlier books in the avoidance of a treatment that is either too brief or too mature. In general, the examples help to make clear the principles under discussion and should help to obviate troublesome points. There is a wide selection of exercises with each new topic—a most desirable asset for this type of text. These are arranged in ascending order of difficulty, problems involving tedious computation being avoided. Answers are included except for those whose numbers are divisible by three. Also at the back of the text is a table of powers and roots, and a table of common logarithms.

This text is, we believe, a teachable one. Missed by this reviewer was any mention of determinants in the chapter on systems of linear equations; historical "asides" that might serve to link material with the rich background of the past; and references to broader or deeper aspects of certain topics that might awaken the interest of the more promising student.—*Paul F. Iverson, Potomac State College of West Virginia University, Keyser, West Virginia.*

Modern Geometry, An Integrated First Course, Claire Fisher Adler (New York: McGraw-Hill Book Company, Inc., 1958). Cloth, xiv+215 pp., $6.00.

This book attempts to bridge "the great instructional gap existing between the completely synthetic Euclidean geometry of the secondary curriculum and abstract modern geometry with its emphasis on algebraic analysis." It is written for a beginning one-semester course in college geometry. The text is divided into three parts: I Foundations and Selected Euclidean Geometry (60 pp.); II Projective Geometry (90 pp.); and III Non-Euclidean and Metric Projective Geometries (48 pp.).

The Foundations (27 pp.) includes a discussion of types of reasoning, a finite geometry, properties of sets of axioms, and Hilbert's axioms. The topics selected from Euclidean geometry include the theorems of Menelaus, Ceva, and Desargues with a few applications; harmonic sets of points and lines; an introduction to cross ratio; and a few theorems regarding inverse points, lines, and circles.

The topics from projective geometry and the scope of the treatment may be indicated by the chapter titles, pages, and the number of exercises. Chapter 6: Projections, invariants, and other unifying concepts (7 pp., 2 ex.). Chapter 7: Basic axioms, duality, Desargues' Theorem, and perspective figures (12 pp., 11 ex.). Chapter 8: Projective theory of harmonic elements, additional axioms (separation, continuity) (10 pp., 7 ex.). Chapter 9: Perspectivities, projectivities, and the projective theory of conics (classification of conics, Pascal's and Brianchon's Theorem) (16 pp., 10 ex.). Chapter 10: Coordinate projective geometry (23 pp., 33 ex.). Chapter 11: Transformation theory (rigid motions, reflections, translations, rotations, groups, similarities) (20 pp., 22 ex.).

In Part III we have: Chapter 12: Early axiomatic theories and the later metric approach to non-Euclidean geometry (13 pp., 9 ex.). Chapter 13: Parabolic (metric) geometry (cross ratio, classification of conics) (9 pp., 7 ex.). Chapter 14: Hyperbolic geometry (14 pp., 11 ex.). Chapter 15: Elliptic geometry (10 pp., 8 ex.).

This reviewer finds the book an interesting and welcome change from the traditional college geometry. There is a serious problem in the selection of topics because of the limitation of time available. Undoubtedly many people would like more advanced Euclidean geometry and many people would like more projective geometry. The restriction to plane geometry is conventional. The reviewer feels that the author's selection of topics is reasonable.

Teachers looking for a new approach in college geometry will want to examine this book. This reviewer feels that it will be very difficult to communicate ideas without more exercises.—*Bruce E. Meserve, Montclair State College, Montclair, New Jersey.*

When was this written?

"Arithmetic is universally taught in schools, but almost invariably as the art of mechanical computation only. The true significance and the symbolism of the processes employed are concealed from the pupil. . . ." *(See page 563 for the answer.)*

• **TIPS FOR BEGINNERS**

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Teach loci with wire and paint

by Donald A. Williamson, Bethesda-Chevy Chase Senior High School, Chevy Chase, Maryland

Several years ago I asked each of my students this question, "Which section of of the plane geometry course did you find the most difficult to understand?" Almost 80 per cent of five different geometry classes had one answer, "Loci problems." The following year I concentrated on teaching this troublesome chapter, using the best techniques I knew, asked the students the same question, and received the same answer. Last year, with the aid of the art department and the metal shop, I was able to use a laboratory approach which practically eliminated this teaching problem.

From brass wire, members of the metal shop class constructed models of each basic locus problem. The nine basic problems used were as follows:

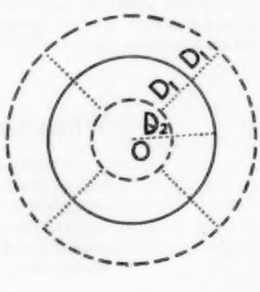
1. Points equidistant from two given points A and B;
2. Points equidistant from two parallel lines L_1 and L_2;
3. Points equidistant from two intersecting lines L_1 and L_2;
4. Points at a given distance D from a given point P;
5. Points at a given distance D from a given line L;
6. Points equidistant from two given concentric circles O and O';
7. Points at a given distance D_1 from a given circle O whose radius is a given distance D_2 when $(D_1 > D_2)$;
8. Points at a given distance D_1 from a given circle O whose radius is a given distance D_2 when $(D_1 = D_2)$;
9. Points at a given distance D_1 from a given circle O whose radius is a given distance D_2 when $(D_1 < D_2)$.

At first glance these problems seem rather elementary. However, the correct statements of the fundamental loci theorems can become rather complicated. For example, problem 9 is stated in theorem form as follows:

> The locus of points at a given distance D_1 from a given circle O with a radius D_2, where $D_1 < D_2$, consists of two circles which are concentric with the given circle and whose radii are $(D_1 + D_2)$ and $(D_2 - D_1)$, respectively.

To make the models more usable, the art class painted the wires different colors. The sections that represented the conditions of the problem were painted black and are shown in this article as solid lines. All wires that showed given distances were silvered, shown here as dotted lines. The bright colors—red, green, yellow, and orange, shown here as dashed lines—depicted the locus or loci for each special case. A model of locus problem 9 is shown in Figure 1. Figure 2 shows a model of locus problem 2.

Figure 1



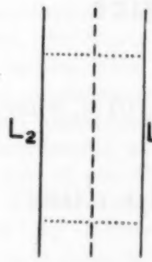


Figure 2

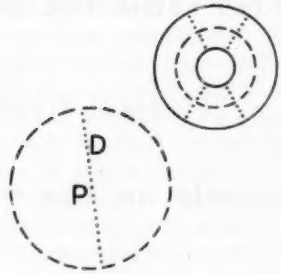


Figure 3

No points that satisfy the two conditions

The models are quite helpful in locating points that satisfy two conditions. During instruction one student can hold one model in a stationary position while another student moves a second model to different positions in the same plane. By noting the points of intersection of the bright-colored wires, pupils can see the various solutions of two sets of conditions. The following example illustrates this idea: What is the locus of points at a given distance D from a given point P and at the same time equidistant from two concentric circles?

The models will show three possible solutions, depending upon the position of point P with respect to the concentric circles. The point or points which meet the given conditions are found only where the two bright-colored lines meet. Figures 3, 4, and 5 illustrate the three possible solutions.

Bright-colored pieces of brass are not guaranteed to solve all the problems in teaching loci. Most of the students, however, will understand the solutions a great deal better because they can see, feel, and experiment with the models. After using the bright-colored models, I found loci far down the list of difficult topics in geometry.

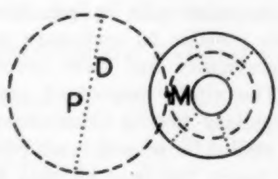


Figure 4

One point, M, that satisfies the two conditions

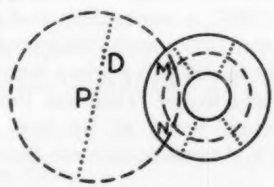


Figure 5

Two points, M and N, that satisfy the two conditions

Answer to "When was this written?"—In 1900, by Nicholas Murray Butler of Columbia University. It can be found in the Editor's Introduction to David Eugene Smith's *The Teaching of Elementary Mathematics* (Toronto, George N. Morang and Company, Limited, 1901).

• NOTES FROM THE WASHINGTON OFFICE

Edited by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

Comments on the new membership dues

New dues are in effect

Undoubtedly all members of the Council have heard by now of the increase in membership and subscription dues that went into effect with the 1958–59 school year. The new dues were first announced in the February 1958 issue of THE MATHEMATICS TEACHER and *The Arithmetic Teacher* and have been used on bills and publicity materials since April.

Why were the new rates needed?

The question is frequently asked as to why an increase in rates was needed. The need arose through sheer financial necessity. When the budget was made in the spring of 1957, it was evident that the old dues at the 1957 membership level could not, in the face of rising prices, support the Council any longer. Therefore the Board of Directors voted at its next official meeting in August to increase the dues.

Was the increase fair in comparison with teachers' salaries?

Several persons have commented that the increase seemed too great in light of the trends in teachers' salaries. The following table gives a fair basis for comparison.

	1947	1958	PER CENT OF INCREASE
Average teacher's salary	$2254	$4650	106
NCTM dues	$3.00	$5.00	66⅔

An increase in dues from $3.00 to $5.00 seems like a big jump if one considers a span of only one year. However, the new dues were required by an accumulation of rising prices covering 11 years. (The previous dues were set in 1947.) Teachers' salaries have more than doubled during this interval of time. But Council dues have gone up only two thirds. Thus it seems that from an economic standpoint the Council has held the line pretty well. Membership in the Council is, from many viewpoints, relatively a much better bargain today than it was 11 years ago.

Will the new dues increase the effectiveness of the Council?

The increased income from the dues should do much more than enable the Council to balance its budget. Already it has been possible to undertake projects which previously had been beyond our reach. The vitally important project of the Secondary School Curriculum Committee, stalled for several years because of lack of funds, has been allotted $10,000. The Elementary School Curriculum Committee has been authorized to spend up to $1000 in planning its program. The purchase of complete Addressograph machinery, to improve service from the Washington Office, has been made possible. A new Membership Directory has been printed.

Have subsequent events justified the increase in dues?

The Financial Report which appeared in the October issue of THE MATHEMATICS TEACHER showed that we got along much better during the past school year than

had been anticipated. Instead of the expected deficit, the Council actually experienced a surplus. This was the direct result of two circumstances which could not have been anticipated: the vigorous growth in membership and the surprisingly good sale of the 23rd Yearbook, *Insights into Modern Mathematics*. If there had been any way of knowing that these two events would occur, the dues increase could conceivably have been postponed for another year.

What about the future?

The new dues coupled with the unexpected good fortune of the 1957–58 year have put the Council in a good financial condition and should make it easier for us to cope with the needs and problems that are likely to arise in the future. This increased power of our organization should make membership more attractive than ever. With the loyal support of our members, we should be entering the most effective and vigorous period of our history.

Registrations at two recent conventions

Below are registration reports of two recent meetings of the Council. Both reports are of special interest in that in each case the attendance was the largest in the history of the Council for a meeting of that type. Both meetings reflected the strong interest being felt nationally in improved methods and curricula in mathematics.

Registrations at the Thirty-sixth Annual Meeting

The National Council of Teachers of Mathematics, Hotel Cleveland, Cleveland, Ohio, April 9–12, 1958

Alabama	5	New Jersey	27
Arizona	1	New Mexico	2
Arkansas	6	New York	120
California	24	North Carolina	4
Colorado	12	North Dakota	4
Connecticut	16	Ohio	1009
Delaware	2	Oklahoma	13
District of Columbia	15	Oregon	4
Florida	14	Pennsylvania	146
Georgia	8	Rhode Island	4
Illinois	158	South Carolina	2
Indiana	59	Tennessee	9
Iowa	12	Texas	13
Kansas	6	Utah	2
Kentucky	8	Vermont	1
Louisiana	11	Virginia	30
Maryland	22	Washington	2
Massachusetts	28	West Virginia	13
Michigan	95	Wisconsin	26
Minnesota	10	Wyoming	1
Mississippi	1	Canada	27
Missouri	20	Foreign	3
Nebraska	6		
New Hampshire	7	Total	2008

Registrations at Annual Joint Meeting of NCTM with NEA

Cleveland, Ohio, June 30, 1958

Alabama	2	Nebraska	1
California	12	Nevada	1
Connecticut	2	New Jersey	5
Delaware	2	New York	4
District of Columbia	4	North Carolina	4
Florida	2	Ohio	57
Georgia	1	Oklahoma	3
Idaho	1	Pennsylvania	16
Illinois	16	South Carolina	1
Indiana	7	South Dakota	1
Iowa	2	Tennessee	3
Kansas	3	Texas	9
Kentucky	8	Utah	2
Louisiana	1	Vermont	1
Maryland	5	Virginia	2
Massachusetts	3	West Virginia	8
Michigan	9	Wisconsin	4
Minnesota	3	Foreign	1
Mississippi	1		
Missouri	5	Total	213

Have you read?

Reese, Mina. "Mathematicians in the Market Place," *The American Mathematical Monthly,* May 1958, pp. 332–343.

This article should be in the hands of all counselors in both high school and college. It points out our needs and brings to light areas many of us have never before considered. Did you know there are 8,000 persons employed as mathematicians, that 900 are on the Ph.D. level, but that there are only 700 American Ph.D's? Did you know that one half the vacancies in mathematics are in applied mathematics? That the quality of mathematics in the university and that in industry is nearly equal? I am sure you know that relatively few teachers of mathematics have ever thus applied it.

The author makes a plea for the mathematician of good personality, co-operative attitudes, practical outlook, and an open mind. She gives many areas of mathematics open to all in both industry and teaching as well as a few pointers on preparation. Call this article to the attention of your counselor.—Philip Peak, *Indiana University, Bloomington, Indiana.*

Wolf, Frank, E. "Why Don't We Produce More Einsteins?" *Phi Delta Kappan,* February 1958, pp. 216–217.

This is a challenging and thought-provoking short article. The author's thesis is that to solve a practical problem requires the projecting of one's mind into an imaginary situation. He believes that the ability to conceptualize is learned and, therefore, educators should teach for maximum attainment of this ability. Children must be taught to project their thoughts beyond their experiences. Only in this way can new factors of knowledge grow out of old.

Mr. Wolf believes the scientific method is only one of many systems and that students should be taught systems of thinking. Through projection into the imaginary we open vast new vistas of mental activity for children.—Philip Peak, *Indiana University, Bloomington, Indiana.*

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Operating Committees (1958–59)

NATIONAL COUNCIL REPRESENTATIVES

Secretary of the Board
Houston T. Karnes, Baton Rouge, La., 1959

AAAS Cooperative Committee on Science and Mathematics
Bruce Meserve, Montclair, N. J., 1960

Educational Advisory Committee to Science Service
Veryl Schult, Washington, D. C., 1959

Conference Organization of the Mathematical Sciences
Howard F. Fehr, New York, N. Y., 1959
John Mayor, Washington, D. C., 1960

United States Commission on Mathematics Instruction
Howard F. Fehr, New York, N. Y., 1960
Eugene Northrop, Chicago, Ill., 1961

AAAS Council
Phillip Jones, Ann Arbor, Mich., 1959
Veryl Schult, Washington, D. C., 1959

EXECUTIVE, 1959

Harold Fawcett, Columbus, Ohio
Phillip Jones, Ann Arbor, Mich.
Philip Peak, Bloomington, Ind.

BUDGET

Jackson Adkins, Exeter, N. H., Chairman, 1960
Agnes Herbert, Baltimore, Md., 1959
Vernon Price, Iowa City, Iowa, 1961

Duties: To prepare the operating budget of the Council for the year 1959–60 and to present it to the Board at its annual meeting in Dallas.

AUDITING

Julia E. Diggins, Washington, D. C., 1959
Ethel H. Grubbs, Washington, D. C., 1960

Duties: To make a general audit of the accounts of the National Council for the year 1958–59 and to provide a report to the Board at its 1959 annual meeting in Dallas.

NOMINATIONS AND ELECTIONS, 1959

Milton Beckmann, Lincoln, Neb., Chairman
Clifford Bell, Los Angeles, Calif.
Charles Butler, Kalamazoo, Mich.
Robert Fouch, Tallahassee, Fla.
Martha Hildebrandt, Maywood, Ill.
Mildred Keiffer, Cincinnati, Ohio
Ann Peters, Keene, N. H.
Myron Rosskopf, New York, N. Y.
Marie Wilcox, Indianapolis, Ind.

Duties: To provide a slate of nominees for the 1959 election.

NOMINATIONS AND ELECTIONS, 1960

Ida Bernhard Puett, Atlanta, Ga., Chairman
Milton Beckman, Lincoln, Neb.
Charles Butler, Kalamazoo, Mich.
Howard Fehr, New York, N. Y.
Kenneth Henderson, Urbana, Ill.
Houston Karnes, Baton Rouge, La.
Albert Linton, Philadelphia, Pa.
Ann Peters, Keene, N. H.
Oscar Schaaf, Eugene, Ore.

Duties: To provide a slate of nominees for the 1960 election.

NOMINATION OF EDITOR FOR THE MATHEMATICS TEACHER, 1959

Kenneth Brown, Washington, D. C., Chairman
Walter Carnahan, Lafayette, Ind.
Mildred Keiffer, Cincinnati, Ohio

Duties: To nominate two persons as editor of THE MATHEMATICS TEACHER, and to present these nominees to the Board at its Christmas meeting in New York.

REPORTING ELECTIONS, 1959

Milton Beckmann, Lincoln, Neb., Chairman
Myrl Ahrendt, Washington, D. C.
Harold Fawcett, Columbus, Ohio

Duties: To certify the results of the annual election and to report these results at the annual business meeting.

Affiliated Groups

Elizabeth Roudebush, Seattle, Wash., Chairman, 1959
Northeastern: Catherine Lyons, Pittsburgh, Pa., 1960
North Central: Virginia Pratt, Omaha, Neb., 1959
Western: Lesta Hoel, Portland, Ore., 1959
Southeastern: Houston Banks, Nashville, Tenn., 1960
Central: Adeline A. Riefling, St. Louis, Mo., 1959
Southwestern: Norma Jones, Oklahoma City, Okla., 1959

Membership, 1959

Mary Rogers, Westfield, N. J., Chairman
Myrl Ahrendt, Washington, D. C.
Marian C. Cliffe, Glendale, Calif.
Mary Rickey, Cedar Rapids, Iowa
Elizabeth Roudebush, Seattle, Wash.
These five to select additional members as needed.

Duties: To suggest ways and means of increasing membership in the Council and to translate these suggestions into action through cooperative work with the Affiliated Groups and the State Representatives.

Place of Meeting

Glenn Ayre, Macomb, Ill., Chairman, 1959
Marguerite Brydegaard, San Diego, Calif., 1959
Forest N. Fisch, Greeley, Colo., 1960
Alice Hach, Racine, Wis., 1960
James Nudelman, Cupertino, Calif., 1959

Duties: To study, plan, and report to the Board concerning the location of Council meetings and to recommend to the Board a planned sequence of convention cities through 1965.

Research

John Kinsella, New York, N. Y., Chairman, 1959
Clark Lay, Los Angeles, Calif., 1959
Kenneth Brown, Washington, D. C., 1961
Howard F. Fehr, New York, N. Y., 1961
J. Fred Weaver, Boston, Mass., 1961

Duties: To promote research in mathematics education, to provide a means of summarizing and publishing significant research in this area, and to plan the program for a Research Section at the annual meeting.

The Mathematics Teacher, 1959

Henry Van Engen, Madison, Wisconsin, Editor, 1959
Irwin Brune, Cedar Falls, Iowa, Asst. Editor
Jackson B. Adkins, Exeter, N. H.
Mildred Keiffer, Cincinnati, Ohio
Z. L. Loflin, Lafayette, La.
Philip Peak, Bloomington, Ind.
Ernest Ranucci, Newark, N. J.
Myron F. Rosskopf, New York, N. Y.

The Arithmetic Teacher, 1959

Ben A. Sueltz, Cortland, N. Y., Editor, 1960
Marguerite Brydegaard, San Diego, Calif.
John R. Clark, New Hope, Pa.

The Mathematics Student Journal, 1959

W. W. Sawyer, Middletown, Conn., Editor, 1961
Arnold Ross, Notre Dame, Ind.
Oscar Schaaf, Eugene, Ore.

Supplementary Publications

L. A. Ringenberg, Charleston, Ill., Chairman, 1959
J. Houston Banks, Nashville, Tenn., 1960
Marguerite Brydegaard, San Diego, Calif., 1961
Edwin Eagle, San Diego, Calif., 1960
Burton Jones, Boulder, Colo., 1961
Margaret Joseph, Milwaukee, Wis., 1959
Jesse Osborn, St. Louis, Mo., 1961
Mildred Keiffer, Cincinnati, Ohio, 1960
Helen A. Schneider, Oak Park, Ill., 1961
Robert Seber, Kalamazoo, Mich., 1959
H. C. Trimble, Cedar Falls, Iowa, 1961
James Ulrich, Arlington Heights, Ill., 1960

Duties: To plan a program of small publications in addition to the three Journals and the Yearbooks, to solicit and evaluate manuscripts received, and to submit to the Publications Board those which seem acceptable for publication.

Publications Board

Glenn Ayre, Macomb, Ill., Chairman, 1959
Clifford Bell, Los Angeles, Calif., Chairman, 1960
Henry Swain, Winnetka, Ill., Chairman, 1961
The Mathematics Teacher, Henry Van Engen, 1959
The Arithmetic Teacher, Ben Sueltz, 1960
The Mathematics Student Journal, W. W. Sawyer, 1961
Supplementary Publications, L. A. Ringenberg, 1959

Duties: To propose and coordinate publication policies of the Council, to make recommendations to the Board concerning major publications, and to report action on minor publications to the President.

Yearbook Planning

Myron Rosskopf, New York, N. Y., Chairman, 1959
Bruce E. Meserve, Montclair, N. J., 1961
Robert E. Pingry, Urbana, Ill., 1960

Duties: To report on status of yearbooks in progress, to recommend topics for future yearbooks, and to suggest possible editors and committee members.

Yearbooks

24th Mathematical Concepts
Phillip Jones, Ann Arbor, Mich., Chairman

Harold Fawcett, Columbus, Ohio
Alice Hach, Racine, Wis.
Charlotte Junge, Detroit, Mich.
Henry Syer, Boston, Mass.
Henry Van Engen, Cedar Falls, Iowa

25th Arithmetic
Foster Grossnickle, Jersey City, N. J., Chairman
Dan Dawson, Stanford, Calif.
Ida Mae Heard, Lafayette, La.
Irene Sauble, Detroit, Mich.
Herbert Spitzer, Iowa City, Iowa
Louis C. Thiele, Detroit, Mich.

26th Evaluation
Donovan Johnson, Minneapolis, Minn., Chairman
Robert Fouch, Tallahassee, Fla.
Glenadine Gibb, Cedar Falls, Iowa
Robert Pingry, Urbana, Ill.
Max Sobel, Fair Lawn, N. J.
Ben Sueltz, Cortland, N. Y.
Fred Weaver, Boston, Mass.

National Association of Secondary School Principals Bulletin, 1959

Myron F. Rosskopf, New York, N. Y., Chairman
Glenn Ayre, Macomb, Ill.
Ida May Bernhard, Austin, Tex.
Robert Fouch, Tallahassee, Fla.
Clark Lay, Los Angeles, Calif.
Eugene Smith, Wilmington, Del.

Elementary School Curriculum

J. Fred Weaver, Boston, Mass., Chairman, 1961
Joyce Benbrook, Houston, Tex., 1960
Laura Eads, New York, N. Y. 1960
Ann Peters, Keene, N. H., 1959
Irene Sauble, Detroit, Mich., 1961
Henry Van Engen, Madison, Wis., 1959

Duties: To develop plans for a study of the elementary school curriculum in mathematics and to propose methods for securing the needed financial support.

Secondary School Curriculum, 1959

Frank B. Allen, La Grange, Ill., Chairman
Jackson B. Adkins, Exeter, N. H.
Howard F. Fehr, New York, N. Y.
A. S. Householder, Oak Ridge, Tenn.
Lottchen Hunter, Wichita, Kan.
Burton W. Jones, Boulder, Colo.
John R. Mayor, Washington, D. C.
Bruce Meserve, Montclair, N. J.
Sheldon Myers, Princeton, N. J.
E. B. Newell, Indianapolis, Ind.
Alfred Putnam, Chicago, Ill.
Elizabeth Roudebush, Seattle, Wash.
Helen M. Walker, New York, N. Y.
Marie Wilcox, Indianapolis, Ind.
Lynwood Wren, Nashville, Tenn.
Magnus Hestenes, Los Angeles, Calif. (Advisor)

Duties: To prepare an interim report based on current questions concerning the mathematics curriculum and to move forward in line with the general purposes of the committee with special emphasis on the junior high school program, a program for those not going to college, and a mathematics program for the gifted.

Teacher Education, Certification and Recruitment

Henry Syer, Boston, Mass., Chairman, 1961
Charles Atherton, Shepherdstown, W. Va., 1959
Kenneth Brown, Washington, D. C., 1960
Dan Lloyd, Washington, D. C., 1961
Robert Kalin, Tallahassee, Fla., 1959
David Page, Urbana, Ill., 1960
Richard Purdy, San Jose, Calif., 1959

Duties: To study the present certification requirements for teachers of mathematics in the respective states including the mathematical background of elementary school teachers and to recommend appropriate policy to the Board of Directors.

Television

Joe Hooten, Tallahassee, Fla., Chairman, 1960
George Anderson, Millersville, Pa., 1960
Emil Berger, St. Paul, Minn., 1961
Lewis Scholl, Buffalo, N. Y., 1959
Sylvia Vopni, Seattle, Wash., 1959
David Wells, Omaha, Neb., 1960

Duties: To study the status of television in the teaching of mathematics and to propose policy to the Board of Directors concerning the use of this medium in mathematics education.

Mathematics for the Talented, 1959

Julius Hlavaty, New Rochelle, N. Y., Chairman
Mary Lee Foster, Arkadelphia, Ark.
Frances Johnson, Oneonta, N. Y.
Glen Vanatta, Indianapolis, Ind.
Robert S. Fouch, Tallahassee, Fla.
Joe Payne, Ann Arbor, Mich.
Harry Ruderman, New York, N. Y.

Duties: To develop plans and proposals designed to improve and strengthen the mathematics education of talented students.

Cooperation with Industry, 1959

Marie Wilcox, Indianapolis, Ind., Chairman
William Glenn, Pasadena, Calif.
Paul Gore, Gary, Ind.
Kenneth Kidd, Gainesville, Fla.
Zeke Loflin, Lafayette, La.
Lauren Woodby, Mt. Pleasant, Mich.
G. A. Rietz, New York, N. Y.

Duties: To publish in the official journals of the Council descriptions of cooperation be-

tween industry and mathematics education and to plan the program for a "cooperation with industry" section at the annual meeting.

INTERNATIONAL RELATIONS

E. H. C. Hildebrandt, Evanston, Ill., Chairman, 1959
Ernest Ranucci, Newark, N. J., 1961
Alfred Putnam, Chicago, Ill., 1961
Veryl Schult, Washington, D. C., 1960

Duties: To extend the hospitality and services of the Council to mathematics teachers and students from the United States who are visiting other countries and to foreign mathematics teachers and students in this country, to cooperate with international organizations in securing an interchange of information concerning mathematics education, and to plan the program for an "International Section" at the annual meeting.

CHRISTMAS MEETING, NEW YORK, 1958

Elizabeth E. Sibley, Brooklyn, N. Y., Chairman
George Grossman, New York, N. Y.
Julius Hlavaty, New Rochelle, N. Y.
Sara Malkin, Ridgewood, N. Y.
Rose M. Roll, New York, N. Y.
Ruth Ruderman, New York, N. Y.
Lester Schlumpf, Flushing, N. Y.

DALLAS CONVENTION, 1959

Arthur Harris, Dallas, Tex., Chairman
Jess Cardwell, Dallas, Tex., Cochairman
J. William Brown, Dallas, Tex.
Clifford Breeding, Dallas, Tex.
J. E. Caldwell, Dallas, Tex.
Allen Cannon, Dallas, Tex.
Lois Crawford, Dallas, Tex.
Elizabeth Dice, Dallas, Tex.
Frances Freese, Dallas, Tex.
W. P. Fulton, Dallas, Tex.
Irvin Hill, Dallas, Tex.
Lorena Holder, Dallas, Tex.
Woodrow Keahey, Dallas, Tex.
W. Ogden Kidd, Dallas, Tex.
Don Matthews, Dallas, Tex.
W. E. McElroy, Dallas, Tex.
Florence St. Clair, Dallas, Tex.
J. Bragg Stockton, Dallas, Tex.
Leo Tisdale, Dallas, Tex.
F. L. Williams (Mrs.), Dallas, Tex.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

EIGHTEENTH CHRISTMAS MEETING

December 29–30, 1958
Sheraton-McAlpin Hotel, New York, New York
Elizabeth Sibley, 18 Stuyvesant Oval, New York 9, New York

THIRTY-SEVENTH ANNUAL MEETING

April 1–4, 1959
Baker Hotel, Dallas, Texas
Arthur W. Harris, 4701 Cole Avenue, Dallas 5, Texas

JOINT MEETING WITH NEA

June 29, 1959
St. Louis, Missouri
M. H. Ahrendt, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NINETEENTH SUMMER MEETING

August 17–19, 1959
University of Michigan, Ann Arbor, Michigan
Phillip S. Jones, Mathematics Department, University of Michigan, Ann Arbor, Michigan

Other professional dates

Forty-eighth Annual Meeting of the Central Association of Science and Mathematics Teachers

November 27–9, 1958
Hotel Claypool, Indianapolis, Indiana
Newton G. Sprague, Indianapolis Public Schools, 1644 Roosevelt Avenue, Indianapolis 18, Ind.

Chicago Elementary Teachers' Mathematics Club

December 8, 1958
Toffenetti's Restaurant, 65 W. Monroe Street, Chicago, Illinois
Romana H. Goldblatt, Burley School, Chicago, Illinois